Exploring the orbital evolution of planetary systems

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The aim of this paper is to encourage the use of orbital integrators in the classroom to discover and understand the long term dynamical evolution of systems of orbiting bodies. We show how to perform numerical simulations and how to handle the output data in order to reveal the dynamical mechanisms that dominate the evolution of arbitrary planetary systems in timescales of millions of years using a simple but efficient numerical integrator. Trough some examples we reveal the fundamental properties of the planetary systems: the time evolution of the orbital elements, the free and forced modes that drive the oscillations in eccentricity and inclination, the fundamental frequencies of the system, the role of the angular momenta, the invariable plane, orbital resonances and the Kozai-Lidov mechanism.

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I. INTRODUCTION

With few exceptions astronomers cannot make experiments, they are limited to observe the universe. The laboratory for the astronomer usually takes the form of computer simulations. This is the most important instrument for the study of the dynamical behavior of gravitationally interacting bodies. A planetary system, for example, evolves mostly due to gravitation acting over very long timescales generating what is called a *secular evolution*. This secular evolution can be deduced analytically by means of the theory of perturbations, but can also be explored in the classroom by means of precise numerical integrators. Some facilities exist to visualize and experiment with the gravitational interactions between massive bodies[1–3]. All of them can show the dramatic dynamical effect due to strong gravitational interactions in a short timescale. But in general they are not devised to study the slow changes that planetary systems exhibit in timescales of million of years or even longer. On the other hand, astronomers have several precise orbital integrators to study the long term dynamics of planetary systems[4,5], but their handling in general require some expertise because they are not thought to be used for educational purposes. In spite of the existence of excellent examples halfway between those extremes[6,7], we prefer to present a simpler numerical integrator which its code can be easily examined and modified. The tool we will use here, ORBE, was designed to be precise and fast in the computation of the long term interactions and of very simple use: only one plain text input file with initial conditions and only one plain text output file with the time evolution of the orbital elements of the interacting bodies. Both files can be edited with the preferred plain text editor and the results can be plotted with the preferred graphic utilities. The code, details of the use of ORBE and several examples can be found at its website[8]. This paper is organized as follows: first in Section II we will introduce a brief explanation on orbital elements and orbital integrators, then in Section III we will show how to discover the fundamental properties of planetary systems through some numerical experiments and finally in Section IV we present the conclusions.

II. ORBITAL ELEMENTS AND ORBITAL INTEGRATORS

In orbital simulations of the Solar System the reference system is usually composed by the plane XY defined by the orbit of the Earth as it was in the year 2000, being the X axis pointing to the J2000.0 dynamical equinox, or Aries point. The Z axis is orthogonal and directed according to the sense of the Earth's orbital revolution around the Sun. The center of the system is usually the Sun which is not inertial due to the attractions by the planets, but this is not a problem in order to represent a relative motion. Of course, the equations of motion are written at first in an inertial frame with origin in the barycenter of the system and then is performed the change of the origin to the Sun[9]. Cartesian positions and velocities are not appropriate for the representation of orbital evolutions because they vary quickly according to the revolutions around the central star. Astronomers use the *orbital elements* instead that are slow varying parameters. The orbital elements emerge assuming that instantaneously the orbit is a conic section. The semimajor axis, a, expressed in astronomical units and eccentricity, e, define size and shape of the orbit respectively, being $a(1-e)$ and $a(1+e)$ the *perihelion* and *aphelion* or minimum and maximum distance to the Sun respectively. The inclination, $0^\circ \leq i \leq 180^\circ$, and longitude of the ascending node, $0^\circ \leq \Omega \leq 360^\circ$, are the two angles that define the orientation of the orbital plane with respect to the reference plane XY. The argument of the perihelion, $0^\circ \leq \omega \leq 360^\circ$, or the longitude of the perihelion, $\varpi = \Omega + \omega$, define the orientation of the axis of the ellipse in the orbital plane. Finally the mean anomaly, $0^\circ \leq M \leq 360^\circ$, defines the position in the orbit, being, for example, $M = 0^\circ$ the perihelion and $M = 180^\circ$ the aphelion. Figure 1 shows the angular orbital elements i, Ω, ω with respect to the reference system XYZ. It is possible to show that in case of a regular evolution a, e, i have small amplitude periodic oscillations and are related to the *action* canonical variables, while M, ϖ, Ω include also variations proportional to the time and are related to the *angle* canonical variables[10]. In the numerical integrations, for each object we must specify the set of initial elements $(a, e, i, \Omega, \omega, M)$, which is equivalent to give (x, y, z, v_x, v_y, v_z), and the mass m in units of solar masses ($M_\odot$). Burns[11] presented a useful analytical deduction of the equations for the time evolution of the orbital elements due to different kinds of perturbations when they are expressed in analytical form. Our approach is completely numerical and restricted only to gravitational perturbations, which in fact dominate the long term evolution of planetary systems.

To obtain the position and velocity at time $t_{i+1} = t_i + \Delta t$ starting from position and velocity at time t_i for a particle under an acceleration $\vec{\alpha}_i$ elementary numerical algorithms usually implemented to resolve general problems are roughly of the type

$$\vec{r}_{i+1} = \vec{r}_i + \vec{v}_i \cdot \Delta t \quad (1)$$

$$\vec{v}_{i+1} = \vec{v}_i + \vec{\alpha}_i \cdot \Delta t \quad (2)$$

For the case of orbital dynamics, Encke[9] proposed a very clever alternative taking into account that the total acceleration acting on each particle, $\vec{\alpha}$, is in fact the addition of a large acceleration due to the Sun, $\vec{\alpha}_s$, plus a small

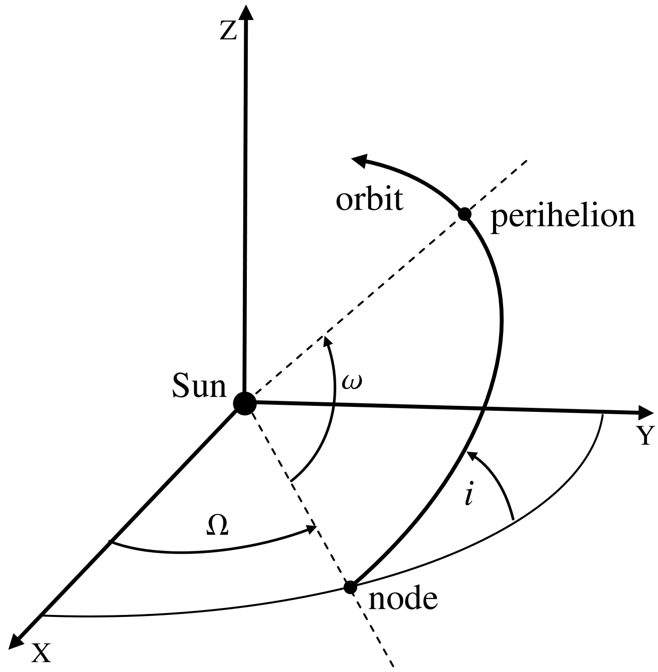

FIG. 1. The angular orbital elements i, Ω and ω. The spatial orientation of the orbital plane is defined by i and Ω and the orientation of the axis of the ellipse in the orbital plane is defined by ω.

acceleration or perturbation, $\vec{\alpha}_p$, due to the other bodies and other eventual non gravitational effects, like the forces generated by emissions jets in comets or the several forces due to solar radiation acting on the surfaces of the bodies[12]. The heliocentric part of the acceleration does not need to be numerically integrated because it has an analytical solution given by the well known two body problem. We only need to integrate the perturbation given by $\vec{\alpha}_p$ which is of the order of 1000 times smaller than the heliocentric acceleration, then we can use timesteps 1000 times greater with the consequent speed up of the numerical integration. A rough scheme for Encke's method is the following. We call S_i the heliocentric, non perturbed, part of the solution defined by the initial conditions $(\vec{r}_i, \vec{v}_i)$:

$$S_i = S(\vec{r}_i, \vec{v}_i) \tag{3}$$

the new position at t_{i+1} is given by S_i evaluated at t_{i+1}:

$$\vec{r}_{i+1} \leftarrow S_i(t_{i+1}) \tag{4}$$

but the velocity at t_{i+1} is the one corresponding to $S_i(t_{i+1})$ plus the perturbation integrated in Δt:

$$\vec{v}_{i+1} \leftarrow S_i(t_{i+1}) + \vec{\alpha}_p(i) \cdot \Delta t \tag{5}$$

Now, the new heliocentric solution at t_{i+1} can be defined:

$$S_{i+1} = S(\vec{r}_{i+1}, \vec{v}_{i+1}) \tag{6}$$

and the procedure continues. Note that the only step that is numerically integrated is the one in Eq. (5) because S is an analytical expression. Another very clever idea is to construct the algorithm such that in each step of the numerical integration the energy is preserved. The *leapfrog* scheme[13], for example, is one of them. These are the key ideas that allowed the astronomers to investigate planetary systems by means numerical integrations of billions of years. We can be confident in these results because using different algorithms elaborated by different researchers the main results are essentially the same. The ORBE integrator is a very simplified version of the code EVORB[14] by Brunini and Gallardo and follows the ideas presented above, uses Encke's method and integrates the perturbations by a leapfrog scheme with a timestep of the order of 1/40 of the smallest orbital period, which is a known good compromise between computation velocity and precision[15]. In its present state ORBE only takes into account the newtonian gravitational interactions, no GR or other forces are considered. Another limitation is that it handles with low precision the events of very close encounters between two orbiting bodies.

III. FUNDAMENTAL PROPERTIES OF PLANETARY SYSTEMS

A. Secular evolution

One of the most impacting results of the planetary theory developed by Lagrange and Laplace in the eighteenth century was the analytical proof of the stability of our planetary system. They showed that in spite of the complexity of the mathematical problem the planetary orbital eccentricities and inclinations have only bounded amplitude oscillations while the semimajor axes are almost constant in time[10]. This can be illustrated integrating with ORBE the Solar System by 1 million years using the initial conditions given in Table I and plotting a, e, i versus time using the output data. Two centuries after the analytical prediction we can verify it numerically in a few minutes.

TABLE I. Initial conditions for the 1 Myr Solar System simulation. Angles in degrees. Data taken from ssd.jpl.nasa.gov.

Planet	a(au)	e	i	Ω	ω	M	mass ($M_\odot$)
Mercury	0.387098	0.205640	7.00	48.32	29.14	143.52	1.660136×10^{-7}
Venus	0.723329	0.006759	3.39	76.66	54.53	24.82	2.447838×10^{-6}
Earth	0.999988	0.016717	0.00	175.41	287.62	257.61	3.040432×10^{-6}
Mars	1.523674	0.093437	1.85	49.54	286.56	226.11	3.227155×10^{-7}
Jupiter	5.202040	0.048933	1.30	100.51	274.18	223.70	9.547919×10^{-4}
Saturn	9.551249	0.054508	2.49	113.63	340.06	38.63	2.858859×10^{-4}
Uranus	19.176263	0.048094	0.77	73.96	98.52	169.39	4.366244×10^{-5}
Neptune	30.098703	0.006894	1.77	131.78	264.31	283.25	5.151389×10^{-5}

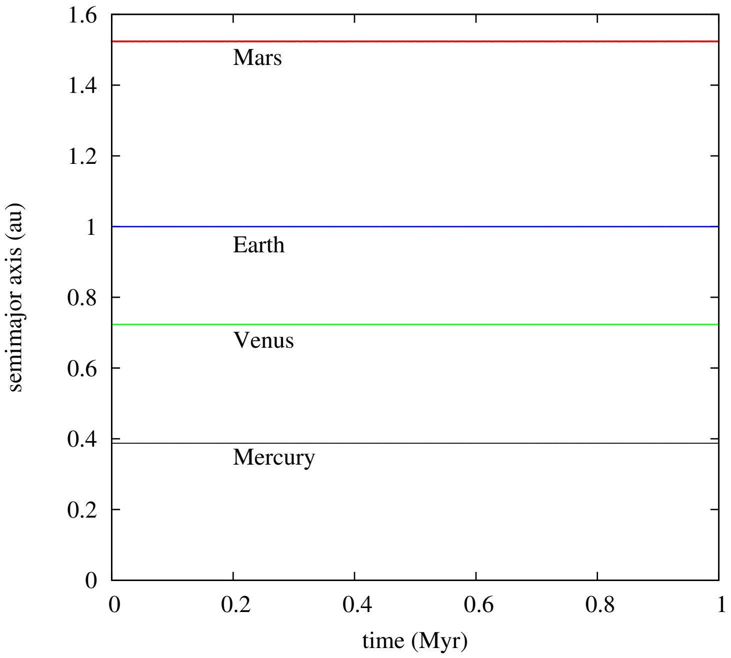

FIG. 2. Time evolution of semimajor axes for the terrestrial planets from present to 1 Myr showing numerically what the theory by Lagrange and Laplace predicted in the eighteen century. This figure was obtained plotting column 3 versus column 1 of the output file orbeout.dat using points. $t = 0$ corresponds to present.

Figure 2 shows the time evolution of the semimajor axes of the four terrestrial planets obtained from the same simulation. No secular trends are observed although there are high frequency oscillations with amplitude of a few 10^{-6} au that cannot be discerned in this scale but can be showed looking at the output data file of the program. Figure 3 shows the time evolution of the eccentricities which seem to be the result of various oscillations. The inclinations show similar behavior to eccentricities. Note that at present Earth's eccentricity is smaller than that of Mars, but this situation was not always the same in the past and will not be always the same in the future. The variations in orbital eccentricity have a known impact in the insolation that a planet receives from the Sun which is one external cause of climate variations[16].

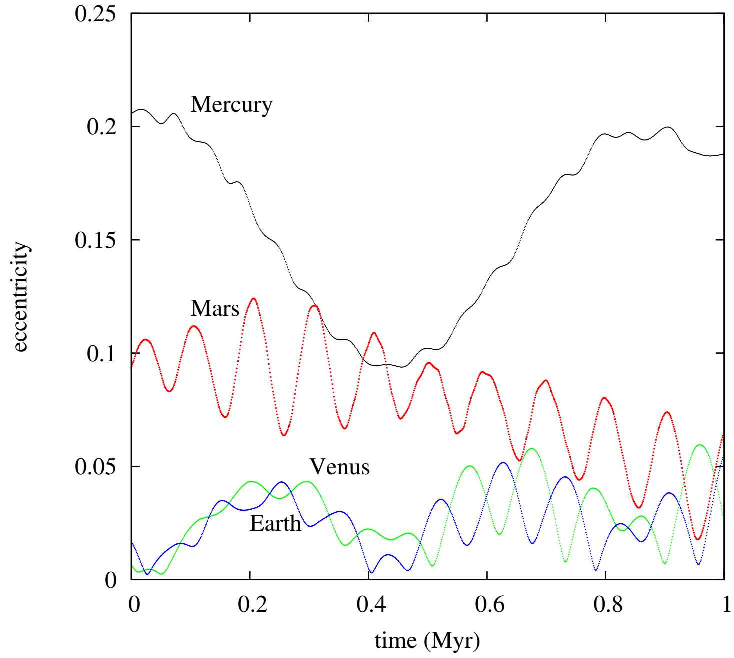

FIG. 3. Time evolution of the eccentricity for the terrestrial planets from present to 1 Myr. This figure was obtained plotting column 4 versus column 1 of the output file orbeout.dat using points.

The time evolution of the system is not altered if we add the dwarf-planets Ceres or Pluto or any asteroid, transneptunian object or comet. If we drop Mercury instead, the outer Solar System is not much affected but changes in $e(t)$ and $i(t)$ for Venus and Earth clearly appear after $\sim 10^5$ years of orbital evolution. Do the experiment by yourself.

B. Angular momentum

It is clear from Fig. 3 that the eccentricities of Earth and Venus evolve in opposed phase. This is more evident for the pair Jupiter-Saturn showed in Fig. 4 using the output of the same numerical integration we have performed. The reason is that these pairs are coupled such that the angular momentum of these subsystems tend to remain constant, as we explain below.

The orbital angular momentum for a planet with mass m is a vector orthogonal to its orbital plane and by definition is the vectorial product $\vec{L} = \vec{r} \times m\vec{v}$ which written in terms of the orbital elements becomes[10]

$$\vec{L} = \frac{mM_\odot}{m + M_\odot}\sqrt{GM_\odot a(1-e^2)}(\sin i \cos\Omega, \sin i \sin\Omega, \cos i) \tag{7}$$

being G the gravitational constant, $M_\odot$ the mass of the Sun and where $mM_\odot/(m + M_\odot) \simeq m$ because $m \ll M_\odot$. Then for the pair Jupiter-Saturn for example, taking into account that they are almost coplanar ($i \sim 0°$) we obtain:

$$L_{JS} \simeq m_J\sqrt{GM_\odot a_J(1-e_J^2)} + m_S\sqrt{GM_\odot a_S(1-e_S^2)} \tag{8}$$

As a_J and a_S remain constant, an increase in e_J must be correlated to a decrease in e_S, which is the effect that we see in Fig. 4. If this reasoning is correct and we run a numerical integration with a "Saturn" in a retrograde orbit, that means orbiting contrary to Jupiter with $i_S \sim 180°$, we will obtain e_J and e_S evolving in phase. The result of this experiment is showed in Fig. 5. Planetary systems composed by only two planets orbiting a star always exhibit coupling between the eccentricities.

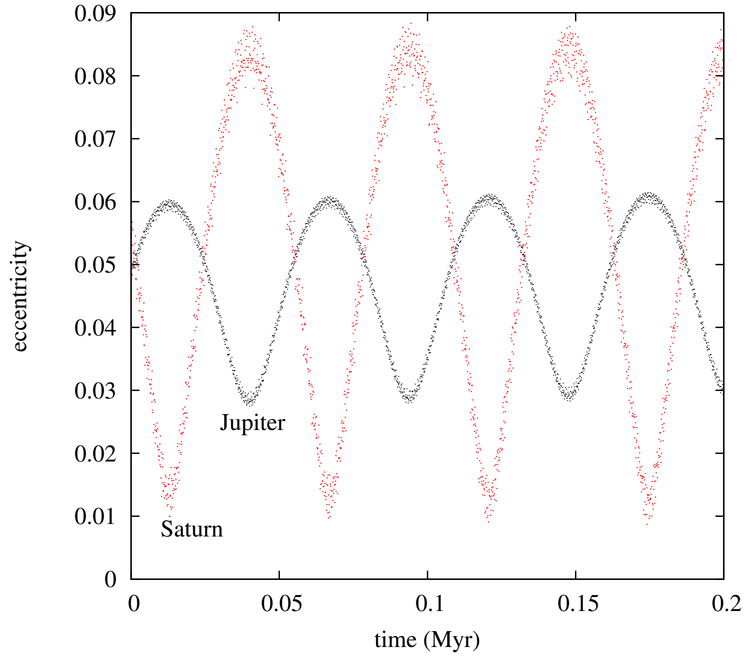

FIG. 4. Coupling between Jupiter and Saturn's eccentricities due to the quasi-conservation of the angular momentum of the subsystem Jupiter-Saturn.

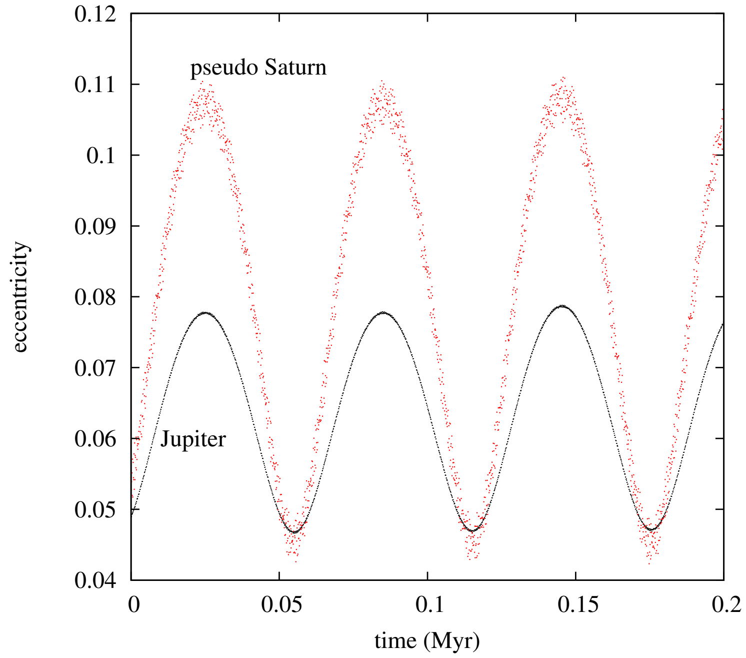

FIG. 5. Coupling between eccentricities of Jupiter and a pseudo Saturn with the same orbital elements of Saturn except its inclination that was taken $i = 178.5°$. In this experiment Jupiter and the pseudo Saturn have their angular momenta with opposed directions.

C. Forced and free modes, fundamental frequencies and chaos

To understand the time evolution of the eccentricities showed in Fig. 3 we will make a simple experiment considering only Jupiter and a fictitious particle ($m = 0$) with initial arbitrary asteroid-like orbital elements given by

$(2.3\text{au}, 0.035, 0.5°, 90°, 270°, 0°)$ and integrate by 40000 years. The time evolution of its eccentricity is showed in Fig. 6. It seems to be a kind of sinusoid but its meaning is revealed when we plot the data in the plane $(k = e \cos \varpi, h = e \sin \varpi)$ obtaining Fig. 7. It is evident that the instantaneous, also called *osculating*, $e(t)$ is the composition of a circulation with almost constant amplitude ($e_{free} \sim 0.012$) which is called *free mode* that circulates with a *proper frequency* around a fixed point at $(0.025, 0.007)$ with modulus $e_{forced} \sim 0.026$, the called *forced mode.* In the plane $(q = i \cos \Omega, p = i \sin \Omega)$ the evolution is also the addition of a constant forced mode and a free circulation[10]. Experimenting with the same asteroid but with different perturbing planets it is possible to explore how the free and forced modes arrange in order to provide the same initial osculating eccentricity.

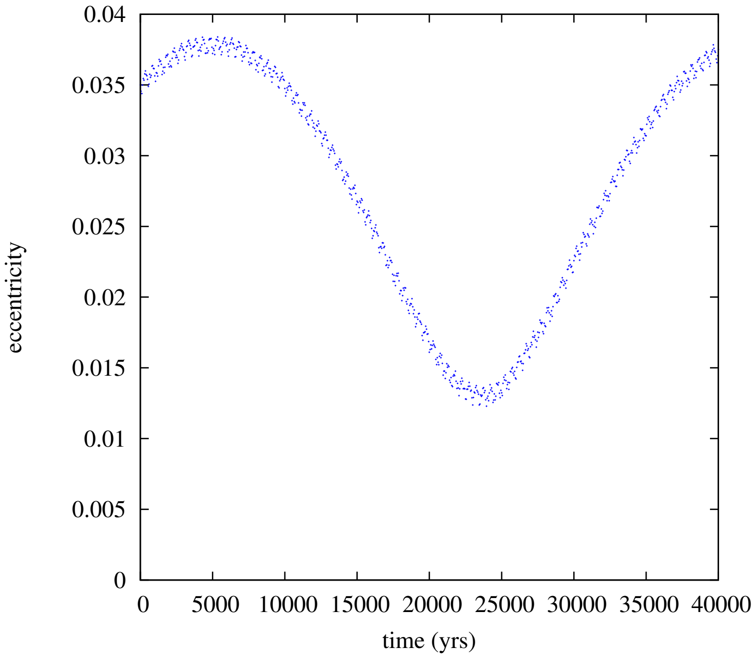

FIG. 6. Time evolution of the eccentricity of a fictitious particle with orbital elements typical of the asteroids perturbed only by Jupiter.

If we consider a system of perturbing planets instead of the simple system composed by the alone Jupiter we will verify that the forced mode is a composition of several vectors each one varying with a different frequency. If we eliminate Jupiter from the Solar System, the component of the forced mode due to Jupiter vanishes and Earth's eccentricity for example will have a smoother time evolution. Figure 8 shows the time evolution of Earth's eccentricity in the actual Solar System and in an hypothetical Solar System in which Jupiter was eliminated.

A spectral analysis of the time evolution of the variables $q(t)$ or $p(t)$ and $k(t)$ or $h(t)$ corresponding to a member of the Solar System will show the presence of a set of well known frequencies, called *fundamental frequencies* of the Solar System and noted as f_i for the variables (q, p) and g_i for the variables (k, h). For example, in Fig. 9 we show the power spectrum of $k(t)$ for the Earth. Three relevant frequencies in units of yr^{-1} appear with large amplitude: 3.3×10^{-6}, 5.7×10^{-6} and 13.5×10^{-6}. They correspond to the fundamental frequencies g_5 due to Jupiter, g_2 due to Venus and g_3 due to Earth[10]. There is another less relevant frequency at 21.8×10^{-6} yr^{-1}, known as g_6 corresponding to Saturn. The existence of these well defined frequencies is a strong indicator of the stability of the system. On the other hand, a planetary system with poorly defined frequencies or varying frequencies is an indicator of its chaotic nature[17]. Several sophisticated tools have been developed to diagnose chaos in the dynamics of the planetary systems. It is not the focus of this paper, but we can have a hint of the magnitude of the chaos in a planetary system looking at the time evolution of the semimajor axes or analyzing how well defined the fundamental frequencies are[18]. In fact, our planetary system has its fundamental frequencies well defined in time scales of millon of years, but in time scales of giga years small variations have been detected[17], which means that in long time scales the Solar System is chaotic, but not necessarily evolving to disruption[19].

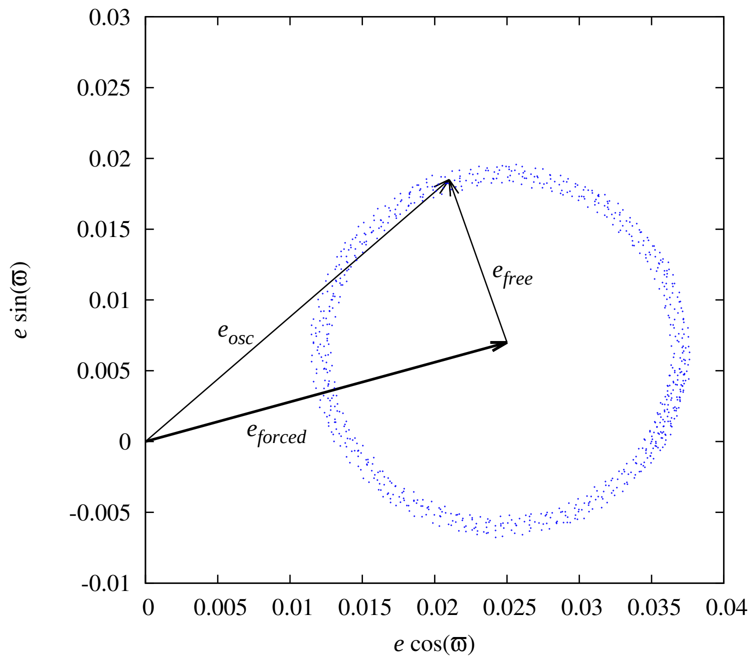

FIG. 7. Same as Fig. 6 but plotting the orbital states in the space $k = e\cos\varpi$ and $h = e\sin\varpi$ showing the forced and free modes in eccentricity. The orbital element ϖ is obtained adding columns 6 and 7 in orbeout.dat.

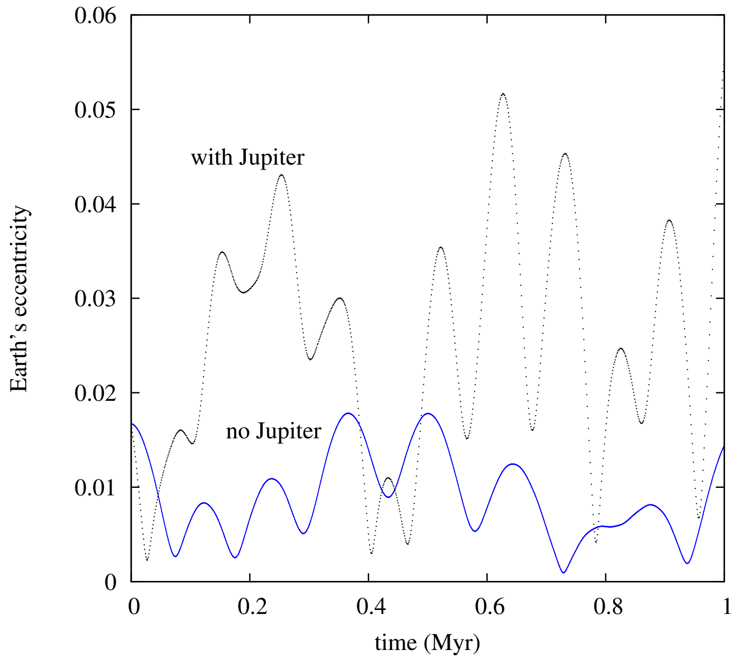

FIG. 8. Earth's eccentricity in a planetary system with and without Jupiter. In the last case the strong forced mode due to Jupiter is not present.

D. The invariable plane

As we have explained in Section II it is a generalized practice to take the reference plane XY coincident with the Earth's orbital plane for the year 2000. That choice is arbitrary from the physical point of view because the Earth's orbit is not dynamically privileged with respect to other orbital planes. There is another plane with a clear physical

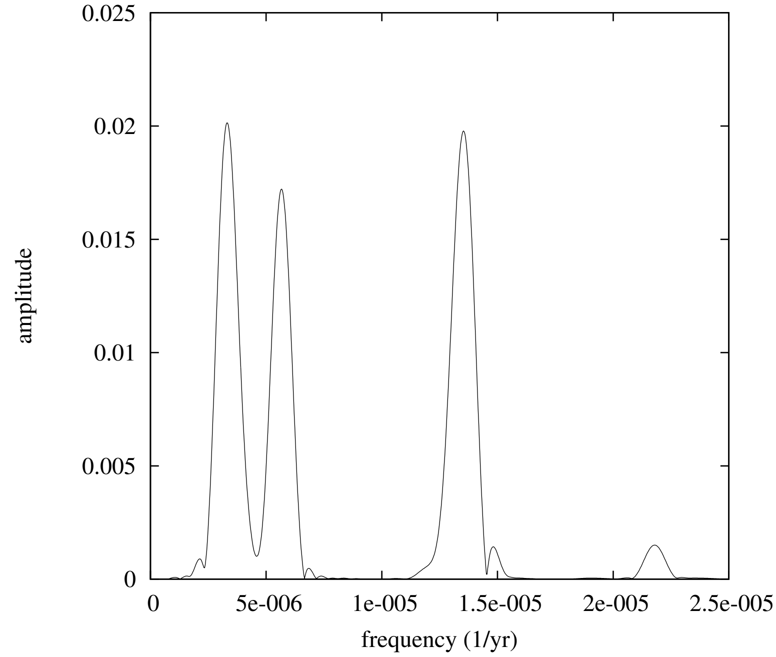

FIG. 9. Power spectrum of the variable $k(t) = e \cos \varpi$ for the Earth showing the most important fundamental frequencies affecting the time evolution of the elements e and ϖ. The amplitude of each line corresponds to the amplitude of the oscillation in e introduced by each frequency. The largest ones corresponds to g_5, g_2 and g_3.

meaning: the *invariable plane*, which is defined as the plane perpendicular to the total angular momentum of the system $\vec{L}_T$:

$$\vec{L}_T = \sum_{i=1}^{8} \vec{L}_i \tag{9}$$

where $\vec{L}_i$ for each planet is given by Eq. 7. In spite of its large mass the contribution due to the Sun can be neglected because is very close to the barycenter and moving with very low velocity. If we consider the system isolated $\vec{L}_T$ will be constant and also the invariable plane. According to Eq. 7 the projection of the angular momentum unity vector for each planet on the reference plane XY is $(\sin i \cos \Omega, \sin i \sin \Omega)$, where we can take i instead of $\sin i$ for low inclination orbits as is the case for the Solar System. If we plot the orbital states in the space $(q = i \cos \Omega, p = i \sin \Omega)$ for all the giant planets along 2 Myr we will obtain Fig. 10. The meaning of that figure is that all planets have their angular momentum unity vectors oscillating around a fixed direction marked with a cross in the figure which is the direction defined by the $\vec{L}_T$ of the system. An analogy can be found between the oscillations of the $\vec{L}$ of a given planet around $\vec{L}_T$ and the precessional motion of a spinning top around the vertical. The invariable plane for planetary systems has its analogue in the horizontal plane for the physics laboratory. For tradition we use the Earth's orbital plane as the reference plane but sometimes for some particular dynamical studies the use of the invariable plane is preferred.

E. Orbital resonances

At present there are around 700.000 asteroids with determined orbital elements in public databases[20], but their distribution between Mars and Jupiter is far from uniform, there are profound gaps and also concentrations. This peculiar distribution in the asteroids' semimajor axes must be due to their different dynamical evolutions. Let's to compute the orbital evolution of a set of 21 fictitious particles, that means $m_i = 0$, distributed with initial semimajor axes between 2.4 and 2.6 au and initial $e_i = 0.1$, $i_i = 5°$ and arbitrary $\Omega_i = \omega_i = M_i = 100°$. We integrate by 1 Myr and take snapshots of the system at 1000 yrs plotting (a, e) for all particles obtaining Fig. 11. Particles with $a_i \sim 2.5$ au are affected by a strong dynamical mechanism, a *mean motion resonance*, that means that these particles have their orbital periods commensurable with the orbital period of a planet, in this case Jupiter. The obvious effect of the resonance is to excite the eccentricities and destabilize the orbits. By the third law of Kepler we know that the

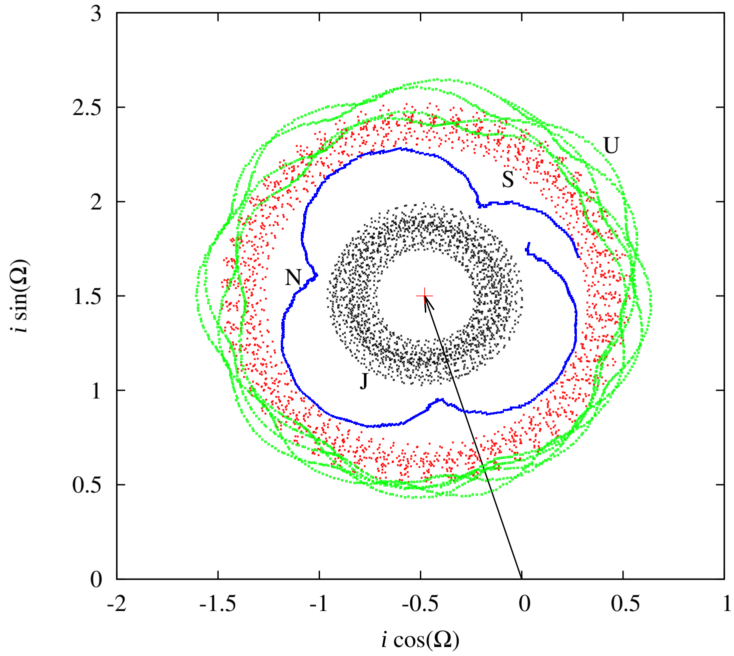

FIG. 10. Evolution in the space $q = i\cos\Omega$ and $p = i\sin\Omega$ of the giant planets from $t_1 = -1$ Myr to $t_2 = +1$ Myr. This plot describes the evolution of the spatial directions of the planetary angular momenta. The trajectory for each planet is labeled with the corresponding letter. The cross indicates the direction of the total angular momentum of the system which is given by $i \simeq 1.6^\circ$ and $\Omega \simeq 108^\circ$.

orbital periods T verify

$$\frac{T_J}{T} = \left(\frac{a_J}{a}\right)^{3/2} \tag{10}$$

Then, in case of a particle with initial $a = 2.5$ au we have $T_J/T = (5.2/2.5)^{3/2} \simeq 3$ and it is said the particle is in the 3:1 resonance with Jupiter. This commensurability in the orbital periods generates a non aleatory spatial distribution of the perturbations by Jupiter making the orbital evolution of the resonant particles different from those non resonant. There is a profound gap at $a \sim 2.5$ au in the semimajor axes distribution of asteroids generated by this resonance, and it is not the only one[10]. The Solar System is full of mean motion resonances but only some of them have dynamical relevance[21]. Some resonances generate unstable dynamics but there are also resonances that provide a very stable orbital evolution. The term resonance is usually associated to a system that is going to disruption, but in orbital dynamics are configurations of equilibrium: sometimes unstable and sometimes stable. The *Hildas*, for example, is a stable dynamical family of asteroids trapped in the 3:2 resonance with Jupiter. In general the stability depends on the amount that the eccentricity grows. Large changes in eccentricity like in the resonance 3:1 make the asteroid to approach to other planets which, by gravitational pull or even collisions, take off the asteroid from the resonance generating the observed gap in the distribution of semimajor axes. A very interesting case of mean motion resonance are the coorbitals, that means, objects with the same orbital period, or trapped in the resonance 1:1. There are several examples of coorbitals in the Solar System and the *quasi-satellites* are probably the most interesting ones[22]. They seem to be satellites of a planet but they are just revolving around the Sun with the exact same planetary orbital period, sometimes ahead of the planet and sometimes behind generating a relative trajectory with respect to the planet that seems to be a satellite-like orbit. This happens not by chance but because is one of the possible configurations of equilibrium in the resonance 1:1. There are precise methods to identify when an asteroid is locked in resonance but a useful indicator is the ratio $(a_P/a)^{1.5}$ acording to Eq. 10, being a_P the semimajos axis of the planet, which must be conserved constant and very close to a simple fraction over some thousands years at least.

Resonances can also occur as a commensurability between the *proper* frequencies of the particle and the *fundamental* frequencies of the planetary system given by the time evolution of the variables $q(t)$ or $p(t)$ and $k(t)$ or $h(t)$. This situation generates a *secular resonance* and large orbital changes usually take place[18]. As the proper frequencies of a particle depends not only on its semimajor axis but also on its eccentricity and inclination, every planetary system has dangerous routes in the space (a, e, i) dominated by secular resonances which could generate drastic orbital changes in asteroids, comets or meteoroids that approach to them.

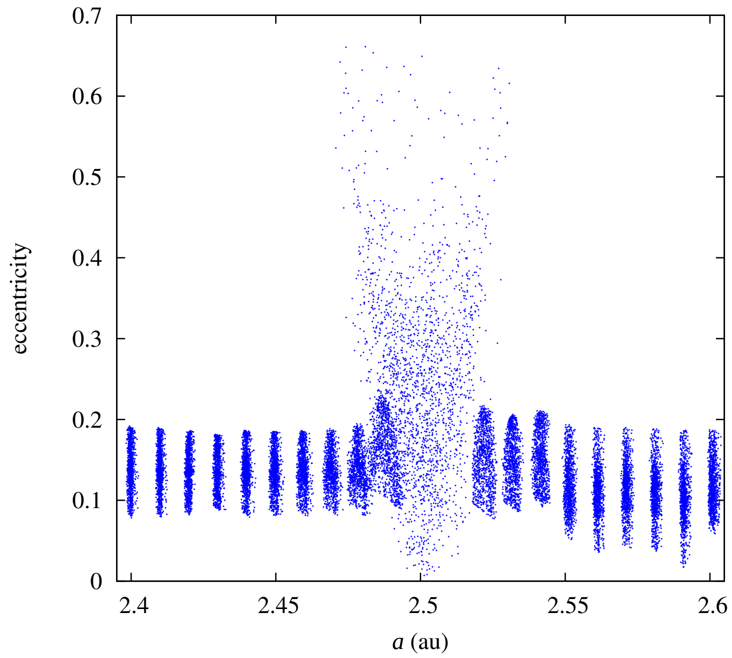

FIG. 11. Evolution of 21 fictitious particles in the space (a, e) during 1 Myr. Particles with $a_i \sim 2.5$ au are strongly affected by the 3:1 resonance with Jupiter increasing their eccentricity up to 0.7. The other particles undergo a bounded secular evolution in eccentricity.

F. The Kozai-Lidov mechanism

Non resonant low eccentricity and low inclination orbits in general will undergo typical secular time evolutions of e and i with small amplitude oscillations without links between them, but for moderate values of eccentricities and inclinations it appears a clear link between e, i and also ω that was first studied by Lidov and Kozai[23]. Its origin is in the quasi conservation of the component L_z of the angular momentum of the body which imposes $\sqrt{a(1-e^2)}\cos i \simeq$ constant according to Eq. 7. In particular, L_z is strictly constant for a particle perturbed by an ideal planetary system composed by coplanar and zero eccentricity orbits[24]. Indeed, in this idealized case the study of the long term evolution can be done as proposed originally by Gauss replacing our planetary system by a system of concentric coplanar rings each one corresponding to one planet, with the radius and mass corresponding to the semimajor axis and mass of the planet respectively. This axisymmetric distribution of mass generates a gravitational force, $\vec{F}$, on a body located at $\vec{r}$ that is not central but is always coplanar with $\vec{r}$ and the axis Z. As the rate of change of the angular momentum equals the applied moment:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \tag{11}$$

it follows that $d\vec{L}/dt$ is orthogonal to Z, or $dL_z/dt = 0$. In the real Solar System the L_z of asteroids or comets is quasi constant.

This mechanism is responsible for very large excursions in e and i driving in some cases to the collision with the central star. In the Solar System those objects are known as *sungrazers*[25]. Thousands of sungrazers comets are known and some asteroids also are in this type of orbit. In Fig. 12 we show the obtained $e(t), i(t)$ for a fictitious particle with initial circular orbit but with $i = 70°$ under the perturbation of the four giant planets. Its semimajor axis remains almost constant but its eccentricity suffers large variations correlated with changes in inclination. It is convenient to follow the evolution of the perihelion distance given by $a(1-e)$ because when it reaches the value $\sim R_\odot \simeq 0.005$ au a collision with the Sun is certain.

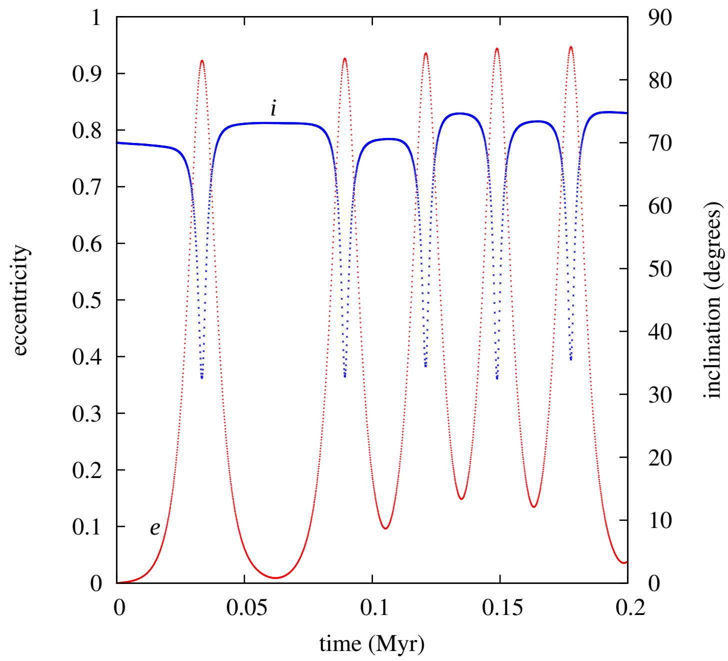

FIG. 12. Kozai-Lidov mechanism for a fictitious particle with initial $e = 0, i = 70^\circ$ perturbed by the four giant planets in their actual orbits. KL mechanism generates large coupled excursions in eccentricity and inclination.

IV. CONCLUSION

By means of numerical simulations using a physical model for a planetary system we will not discover new fundamental laws of physics but we can discover the emerging physical properties of complex dynamical systems. These properties can be deduced by sophisticated perturbative theories but can also be observed by means of a well conducted analysis of the output of the numerical integrations. We have showed some of the basic dynamical properties of the planetary systems illustrating with the Solar System but these properties can also be found in extrasolar[26] and fictitious systems. Several experiments can be performed with a code like the one we have presented in order to explore and reveal the dynamics of planets and minor bodies. Some examples can be found at the ORBE website and the initial conditions for the examples presented here can also be found there. We remark that as ORBE is only for educational purposes, serious research must be done using other professional integrators. A planetary system is usually imagined as a static collection of ellipses but a very rich dynamics emerges when we consider their mutual interactions. Regular oscillations with well defined frequencies or chaotic evolutions that sometimes led to catastrophic ends are two well differentiated dynamical evolutions. As the observed planetary systems are in general the result of billions of years of orbital evolution it is largely more probable to find stable planetary systems than unstable ones. Nevertheless, unstable planetary systems can be found if they are in the first steps of their orbital evolution, like the ones that exhibit traces of the accretion disk. It is exciting that we can explore them in the classroom with simple numerical integrators.

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