Chapter 14

The Moon and its Rotation

14.1 Cassini's Laws

It is well known that the Moon, roughly speaking, always points one face toward the Earth; as we shall see, this leads to a restriction on the possible figure of the Moon. This, and other regularities in the rotation of the Moon, are described in three laws due to Cassini. These laws have been confirmed observationally, but they are a simplification of the actual state of affairs, since there are additional small oscillations about the regular motion that they predict. The laws, together with these additional oscillations (when they can be observed), furnish data about the gravitational field of the Moon.

To consider Cassini's laws, let the Moon be at the center of the celestial sphere. Let Π_1 be the plane of the Earth's orbit around the Moon, with pole P. Let Π_2 be the plane through the Moon parallel to the ecliptic, with pole Z. Let Π_3 be the plane of the Moon's equator, with pole C. (See Figure 14.1). The three laws are:

1. The Moon has uniform rotation about an axis fixed in the Moon, the period of rotation being equal to the sidereal period of the Moon.

2. The angle CZ is constant; about $1°35'$.

3. While the node of the Moon's orbit regresses, its inclination remains constant; about $5°9'$. As the node moves, the arc of the great circle PC always contains Z. This latter condition implies that the three planes Π_1, Π_2, and Π_3 meet in a line, and that Π_2 lies between Π_1 and Π_3.

In the following sections we shall establish these laws by showing that the oscillations about the steady motion that they predict are stable, subject to certain restrictions on the field of the Moon.

14.2 The Eulerian Equations

Let the principal axes of inertia of the Moon be the axes defining the coordinates (x, y, z). One of these must, by symmetry, nearly point toward the

Earth E; if G is the center of mass of the Moon (and therefore the origin of coordinates), let this axis be GA, or the x-axis. Also, one axis must point in nearly the same direction as the axis of rotation of the Moon; let this be GC, or the z-axis. With respect to these axes, let GZ and GE have direction cosines (p, q, r) and (λ, μ, ν), respectively; then p, q, μ, and ν are small quantities (the reader should confirm this and find out just how small they are) whose squares will be neglected subsequently; also r and λ are nearly equal to one.

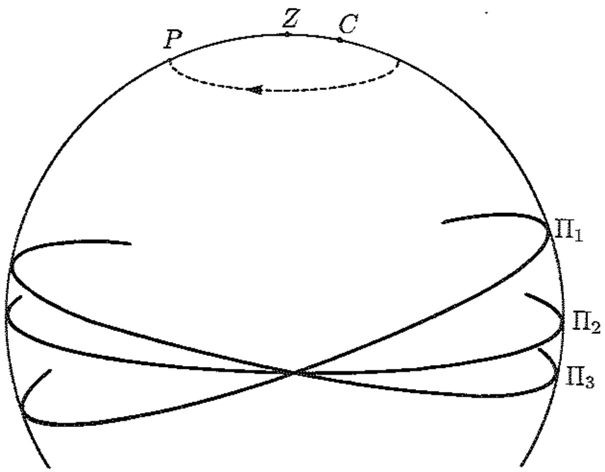

Figure 14.1

The couple exerted by the Earth on the Moon is small, and, in evaluating it, it is sufficient to assume that the Earth is a sphere and that it is a constant distance R from G. Let M be the mass of the Earth, and let A, B, and C be the principal moments of inertia of the Moon. Let the angular velocity of the Moon have components ω_x, ω_y, and ω_z about the x-, y-, and z- axes; the first two are small, and their squares and product will be neglected.

Euler's equations are, in full,

$$A\dot{\omega}_x - (B - C)\omega_y\omega_z = \frac{3MG}{R^3}(C - B)\mu\nu,$$
$$B\dot{\omega}_y - (C - A)\omega_z\omega_x = \frac{3MG}{R^3}(A - C)\nu\lambda,$$
$$C\dot{\omega}_z - (A - B)\omega_x\omega_y = \frac{3MG}{R^3}(B - A)\lambda\mu.$$

Neglecting second-order small quantities, we find

$$A\dot{\omega}_x - (B - C)\omega_y\omega_z = 0, \tag{14.2.1}$$

$$B\dot{\omega}_y - (C - A)\omega_z\omega_x = \frac{3MG}{R^3}(A - C)\nu, \tag{14.2.2}$$
$$C\dot{\omega}_z = \frac{3MG}{R^3}(B - A)\mu. \tag{14.2.3}$$

14.3 The Libration in Longitude

In the following sections we shall be concerned with the ***physical*** (as opposed to the *optical*) libration of the Moon. We note that if Cassini's laws held exactly, then there would be no physical libration. In this section we shall consider equation (14.2.3); this involves ω_z only, and so can be treated separately.

Let the line joining the Earth and Moon make an angle θ with some fixed direction, and let $\angle AGE = \phi$. (See Figure 14.2.) ϕ is a measure of the optical libration in longitude and is never more than about 7°; we will, therefore, feel justified in neglecting its square. The optical libration in latitude is also small; neglecting its square, we can say that GA makes an angle $(\theta + \phi)$ with the fixed direction. From Figure 14.2 we see that a positive ω_z would make $(\theta + \phi)$ decrease; hence

$$\omega_z = -\frac{d}{dt}(\theta + \phi).$$

Now

$$\mu = \cos\angle BGE = \sin\phi = \phi,$$

to the order of accuracy considered; so (14.2.3) becomes

$$\frac{d^2}{dt^2}(\theta + \phi) = -\frac{3GM}{R^3}\frac{B - A}{C}\phi. \tag{14.3.1}$$

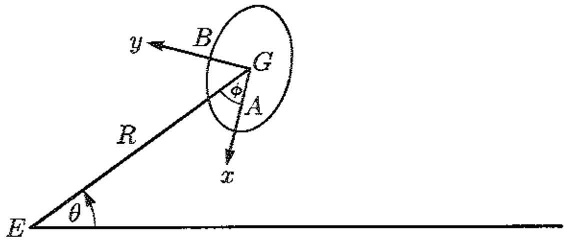

Figure 14.2

Consider first the case when the orbit of the Earth around the Moon is circular, so that $d^2\theta/dt^2 = 0$. ϕ must always remain small, so that (14.3.1) describes simple harmonic motion and must be of the form

$$\frac{d^2\phi}{dt^2} + w^2\phi = 0. \tag{14.3.2}$$

This requires

$$w^2 = \frac{3MG}{R^3}\frac{B-A}{C}. \tag{14.3.3}$$

Then $B > A$, and for this to be so, $a > b$, where a and b are the lengths of the intercepts of the x- and y-axes with the Moon. The solution to (14.3.2) is

$$\phi = \alpha\cos(wt+\beta), \tag{14.3.4}$$

where α and β are arbitrary constants. If this oscillation could be observed, then its period would lead to a value for $(B-A)/C$; but it is too small to be detected.

Next we consider the consequences of the fact that the Moon's orbit is not a circle. If Cassini's first law were accurate, then

$$\dot\theta + \dot\phi = n, \quad \text{a constant,}$$

and

$$\theta + \phi = nt + \epsilon.$$

If t is measured from perigee, then ϵ is a small quantity of the same order as ϕ; θ is the true anomaly. In order to allow for periodic deviations from Cassini's law, put

$$\theta + \phi + \chi = nt + \epsilon, \tag{14.3.5}$$

so that (14.3.1) becomes

$$\ddot\chi = -w^2(\chi + \theta - nt - \epsilon)$$

or

$$\ddot\chi + w^2\chi = -w^2(\theta - nt - \epsilon). \tag{14.3.6}$$

Now $(\theta - nt)$ is the equation of the center for a Keplerian orbit and can be expanded accordingly. But we must allow for the fact that the orbit of the Moon is not even Keplerian. Then the right-hand side of (14.3.6) can be expanded in a series

$$\sum H\sin(ht+h'),$$

which will include all the periodic inequalities in the Moon's motion. (We neglect the secular acceleration of the Moon.) To solve (14.3.6), consider the equation

$$\ddot\chi + w^2\chi = -w^2 H\sin(ht+h').$$

The complementary function is $\alpha\cos(wt+\beta)$, and a particular integral is of the form

$$X\sin(ht+h'),$$

where, by substitution, we find

$$X = \frac{w^2 H}{w^2 - h^2}.$$

Hence the solution is

$$\chi = \alpha\cos(wt+\beta) + w^2\frac{H}{w^2-h^2}\sin(ht+h').$$

Then the complete solution of (14.3.6) is

$$\chi = \alpha\cos(wt+\beta) + w^2\sum\frac{H}{w^2-h^2}\sin(ht+h'). \tag{14.3.7}$$

We have seen that the complementary function, giving the free oscillations, is too small to be observed. The terms in the particular integral are the forced oscillations; for any one to be appreciable, it must have large H or small h. The two most promising inequalities are the elliptic inequality (the first term in the expansion of the equation of the center), for which

$$H = 22639''.1 \quad \text{and} \quad \frac{2\pi}{h} = \text{the anomalistic month,}$$

and the annual equation, for which

$$H = -668''.9 \quad \text{and} \quad \frac{2\pi}{h} = \text{the anomalistic year.}$$

Now, $(B-A)/C$ is very small, and so, therefore, is w; and it is the annual equation that provides the larger forced oscillation. This can be observed, although not very accurately, and from observations of its amplitude the value of $(B-A)/C$ can be found.

We have now shown that Cassini's first law is true in the sense that oscillations about the state that it describes are small and stable, provided $a > b$. We shall next establish the other two laws in the same way.

14.4 Other Oscillations

In Figure 14.3, the dotted line represents the path of the Earth on the celestial sphere (with the Moon at it center), and N is its ascending node. The great circle through Z and E cuts Π_2 at H. Then $EH = l$ is the latitude of E. Neglecting the effects of precession on the position of ♈, we can write

$$♈N = -gt + \text{constant},$$

where g is the rate of regression of the nodes. Let the orbit of the Earth be inclined at an angle k to Π_2; in the course of the time interval t, N will regress through an angle $gt\cos k$ with respect to the Earth's orbit; but k is small, so this may be taken as gt.

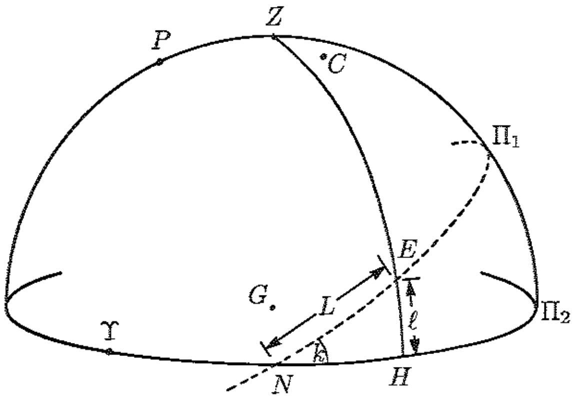

Figure 14.3

We have finished with equation (14.2.3). In considering equations (14.2.1) and (14.2.2) we need not concern ourselves with variations in ω_3, which we shall now take to be n; nor are we interested in μ or ϕ; so we can assume that the Earth moves in a circular orbit with angular motion n. Then

$$NE = L = nt + gt + \gamma,$$

where γ is a constant. Now

$$\begin{aligned}\cos ZE &= p\lambda + q\mu + r\nu \\ &= p + \nu,\end{aligned}$$

approximately. Also

$$\begin{aligned}\cos ZE &= \sin l \\ &= \sin k \sin L \\ &= k \sin L,\end{aligned}$$

approximately. Hence

$$\left.\begin{aligned}\nu &= k\sin L - p \\ &= k\sin(nt + gt + \gamma) - p.\end{aligned}\right] \qquad (14.4.1)$$

Before solving the equations of motion, we shall find the condition for P, Z, and C to lie on a great circle, as required by Cassini's third law. N is the pole

of the great circle PZ, so if C lies on this circle, GN must be perpendicular to GC. In the work following, $\hat{\mathbf{C}}$ will stand for the direction of $\mathbf{GC}$, and so on. Cassini's third law requires

$$\hat{\mathbf{C}} \cdot \hat{\mathbf{N}} = 0.$$

To find $\hat{\mathbf{N}}$, resolve along $\hat{\mathbf{H}}$ and $\hat{\mathbf{H}} \times \hat{\mathbf{Z}}$, so that

$$\hat{\mathbf{N}} = \hat{\mathbf{H}} \cos L + \hat{\mathbf{H}} \times \hat{\mathbf{Z}} \sin L,$$

to the required order of accuracy, since we are putting $\cos k = 1$. Now

$$\begin{aligned}\hat{\mathbf{H}} &= \frac{\hat{\mathbf{Z}} \times (\hat{\mathbf{Z}} \times \hat{\mathbf{E}})}{\sin ZE} \\ &= \frac{\hat{\mathbf{Z}} \cos ZE - \hat{\mathbf{E}}}{\sin ZE} \\ &= \hat{\mathbf{Z}} \cos ZE - \hat{\mathbf{E}},\end{aligned}$$

approximately, since $\sin ZE = \cos l$, and l is small. Hence

$$\hat{\mathbf{N}} = \cos L(\hat{\mathbf{Z}} \cos ZE - \hat{\mathbf{E}}) + \hat{\mathbf{Z}} \times \hat{\mathbf{E}} \sin L,$$

so that

$$\begin{aligned}\hat{\mathbf{C}} \cdot \hat{\mathbf{N}} &= \cos L(r \cos ZE - \nu) + \sin L(p\mu - q\lambda) \\ &= \cos L(\cos ZE - \nu) - q \sin L,\end{aligned}$$

approximately. But $\cos ZE = p + \nu$; so

$$\hat{\mathbf{C}} \cdot \hat{\mathbf{N}} = p \cos L - q \sin L. \qquad (14.4.2)$$

Let d/dt represent rate of change with respect to the moving axes GA, GB, and GC; then, since $\hat{\mathbf{Z}}$ is fixed in space

$$\frac{d\hat{\mathbf{Z}}}{dt} + \boldsymbol{\omega} \times \hat{\mathbf{Z}} = \mathbf{0}.$$

Resolving, we find

$$\begin{aligned}\dot{p} - \omega_z q + \omega_y r &= 0, \\ \dot{q} - \omega_x r + \omega_z p &= 0, \\ \dot{r} - \omega_u p + \omega_x q &= 0,\end{aligned}$$

or, to the first order,

$$\left.\begin{aligned}\dot{p} - nq + \omega_y r &= 0, \\ \dot{q} - \omega_x r + np &= 0, \\ \dot{r} \qquad\quad &= 0.\end{aligned}\right] \qquad (14.4.3)$$

Differentiating the first two of these, and using the third, we find

$$\left.\begin{aligned}\dot\omega_y &= -\ddot p + n\dot q,\\ \dot\omega_x &= \ddot q + n\dot p,\end{aligned}\right] \tag{14.4.4}$$

since $r \sim 1$.

We can now substitute for ω_x and ω_y and their derivatives in (14.2.1) and (14.2.2) when we find

$$\ddot q + n\dot p - \left(\frac{B-C}{A}\right) n(nq - \dot p) = 0,$$

$$-\ddot p + n\dot q - \left(\frac{C-A}{B}\right) n(\dot q + np) = -\frac{3MG}{R^3}\left(\frac{C-A}{B}\right)(k\sin L - p).$$

Put

$$\frac{C-B}{A} = \sigma \quad \text{and} \quad \frac{C-A}{B} = \tau;$$

then, since $MG/R^3 = n^2$, the equations are

$$\ddot q + n\dot p(1-\sigma) + n^2\sigma q = 0, \tag{14.4.5}$$

$$-\ddot p + n\dot q(1-\tau) - 4n^2\tau p = -3n^2\tau k\sin L. \tag{14.4.6}$$

The solution of these equations will consist of a complementary function, representing free oscillations, and a particular integral, emanating from the term in $\sin L$, giving the forced oscillations. The free oscillations must be stable, and we shall now find the conditions for this to be so.

Put

$$p = Fe^{st}, \quad \text{and} \quad q = Ge^{st};$$

then substitution into (14.4.5) and (14.4.6) gives

$$Gs^2 + nsF(1-\sigma) + n^2\sigma G = 0,$$
$$Fs^2 - nsG(1-\tau) + 4n^2\tau F = 0,$$

since we are ignoring the term in $\sin L$. Eliminating the ratio F/G, we find

$$(s^2 + n^2\sigma)(s^2 + 4n^2\tau) + n^2s^2(1-\sigma)(1-\tau) = 0,$$

or

$$s^4 + s^2n^2(1 + 3\tau + \sigma\tau) + 4n^4\sigma\tau = 0.$$

Consider this as a quadratic for s^2. For the oscillations to be stable, the two roots must be real and negative. The condition for them to be real is

$$(1 + 3\tau + \sigma\tau)^2 - 16\sigma\tau > 0,$$

which is certainly true. For the roots to be negative, their sum must be negative, or

$$-n^2(1 + 3\tau + \sigma\tau) < 0,$$

which is true, and their product must be positive, or

$$\sigma\tau > 0;$$

i.e.,

$$(C-A)(C-B) > 0.$$

So C is either greater or less than both A and B; therefore the axis along Gz, of length c, is either less or greater than both a and b. Hence the axis of rotation is either the longest or the shortest axis of the Moon.

Now consider the forced oscillations. Try a solution

$$p = P\sin L, \quad q = Q\cos L.$$

By substitution (remembering that $dL/dt = n + g$), we find

$$\left.\begin{aligned} Q\left\{-(n+g)^2 + n^2\sigma\right\} + Pn(n+g)(1-\sigma) &= 0,\\ P\left\{(n+g)^2 - 4n^2\tau\right\} - Qn(n+g)(1-\tau) &= -3n^2\tau k.\end{aligned}\right] \tag{14.4.7}$$

Solving for Q/P, we find

$$\begin{aligned}\frac{Q}{P} &= \frac{n(n+g)(1-\sigma)}{(n+g)^2 - n^2\sigma}\\ &= \frac{\left(1+\frac{g}{n}\right)(1-\sigma)}{1 + 2\frac{g}{n} - \sigma}\\ &= \left(1 + \frac{g}{n} - \sigma\right)\left(1 - 2\frac{g}{n} + \sigma\right)\\ &= 1 - \frac{g}{n},\end{aligned} \tag{14.4.8}$$

where products of small quantities have been ignored at each stage. Similarly we find

$$P = \frac{3nk\tau}{3n\tau - 2g}. \tag{14.4.9}$$

Now

$$\begin{aligned} CZ^2 &= p^2 + q^2\\ &= P^2\left(\sin^2 L + \left(\frac{Q}{P}\right)^2\cos^2 L\right)\\ &= P^2\left(1 - \frac{g}{n} - \frac{g}{n}\cos 2L\right).\end{aligned} \tag{14.4.10}$$

Hence, apart from a small oscillation with period half a nodical month, CZ is constant. This proves Cassini's second law; we shall return to it in a moment, after proving the third law.

From (14.4.2),

$$\hat{\mathbf{C}}\cdot\hat{\mathbf{N}} = (P-Q)\cos L\sin L = \frac{Pg}{2n}\sin 2L. \tag{14.4.11}$$

This oscillates with small amplitude about zero, so that, apart from this oscillation, it is zero, and P, Z, and C lie on a great circle. This also follows if the complementary functions are included in the evaluation of $\hat{\mathbf{C}}\cdot\hat{\mathbf{N}}$.

It remains to find the condition for Z to lie between P and C. Consider the situation when we put $L = 90°$ in the solution; then the Earth E lies on the great circle PZC, and we have (see Figure 14.4)

$$EP > EZ.$$

If the x-axis cuts the celestial sphere at A, then we know that A very nearly lies on the great circle $PZCE$, and since $AC = 90°$, the condition that Z should lie between P and C is

$$AZ > 90°.$$

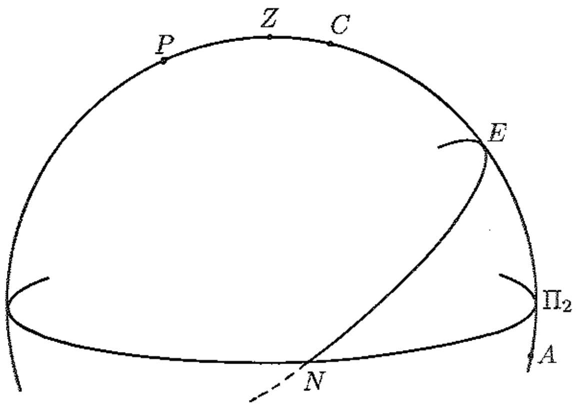

Figure 14.4

Therefore $\cos AZ$ is negative; but $\cos AZ = p = P\sin 90°$, so that

$$P = \frac{3k\tau}{3\tau - 2\frac{g}{n}} < 0.$$

It will be established in a moment that τ is numerically less than $2g/3n$, so that we must have

$$\tau > 0,$$

as the condition for Z to lie between P and C. But this implies

$$C - A > 0,$$

or

$$c < a.$$

Hence c is the shortest of the three axes, and we have

$$a > b > c.$$

Now

$$CZ = |P|\left(1 - \frac{g}{2n}\right),$$

ignoring oscillations. P is small, so we can take

$$CZ = |P| = \left|\frac{3k\tau}{3\tau - 2\frac{g}{n}}\right|. \tag{14.4.12}$$

Substituting the observed values

$$CZ = 91'.4, \quad k = 308'.7, \quad \text{and} \quad \frac{g}{n} = 0.00402,$$

we find

$$\tau = 0.00061.$$

14.5 Problems

1. Find the amplitudes of the two forced oscillations discussed in Section 14.3.

2. Find the periods of the free oscillations of (14.4.5) and (14.4.6).

3. Write down the equation for the variation of longitude, taking into account the secular acceleration of the Moon. Discuss the solutions and the orders of magnitude of any modifications to the earlier theory.

4. If the center of mass of the Moon is constrained to describe a circle, with a uniform angular velocity n about a fixed center of force O attracting according to Newton's law, show that the axis GA of the Moon will oscillate on each side of GO or will make complete revolutions relative to GO, according as the angular velocity of the Moon about its axis at the moment when GA and GO coincide in direction is less or greater than $(n + w)$, where w has the meaning given to it in Section 14.3. Find also the extent of the oscillations.

5. A body free to turn about its center of gravity, which is fixed, is in stable equilibrium under the attraction of a distant fixed particle M' at distance R. Show that the axis of least moment is turned toward the particle. Show also that the times of the principle oscillations are

$$2\pi\left\{\frac{BR^3}{3M'G(C-A)}\right\}^{1/2} \quad \text{and} \quad 2\pi\left\{\frac{CR^3}{3M'G(B-A)}\right\}^{1/2}$$

If the body be the Earth, and M' the Sun, show that the smaller of these two periods is about ten years.

Appendix A

Properties of Conics

A.1 General Properties

The word "conic"comes from the phrase "conic section." Suppose that we have a circle and a point V which is not in the plane of the circle; then lines containing V and points on the circle describe a cone. Any plane section of this cone will be a conic, and any conic can be constructed in this way. (See Figure A.1.)

The graph of the general equation of second degree in cartesian coordinates is (if there are real points satisfying the equation) a conic. Conics can be defined in this way, and their properties discussed through the properties of quadratic forms. One property is that by a transfer of origin and a rotation of the axes, we can *usually* reduce the equation to the form

$$ax^2 + by^2 + c = 0. \tag{A.1.1}$$

If this is possible (one exception is the parabola) then the origin of coordinates is called the *center* of the conic, and the coordinate axes are the *axes* of the conic. For work in cartesian coordinates it is usually simplest to use these axes.

Another important definition can be derived as follows. Pick one of the conic sections from Figure A.1 and insert between the section and the vertex the largest possible sphere; this will touch the cone in a circle, C. Let the plane containing the circle be called Π. The line of intersection of Π and the plane of the conic section is called a *directrix* of the conic. Let the sphere touch the plane of the conic section at a point S; this is a focus of the conic. See Figure A.2.

Let P be a point on the conic, and let the line VP intersect the circle C at the point A. The lines PS and PA touch the sphere at S and A, and therefore $PS = PA$. Now drop a perpendicular from P onto the plane Π, meeting the plane at Q. Let the *half*-angle of the cone be θ. Then from the triangle PQA we have $AP = PQ\sec\theta$. Let the angle between Π and the plane of section be ϕ, and let R be a point on the directrix such that PR is perpendicular to