Chapter 8

The Three-Body Problem

8.1 The Restricted Three-Body Problem — Jacobi's Integral

The general problem of the motion of three bodies (assumed to be point masses), subject only to their mutual gravitational attractions has not been solved, although many particular solutions have been found. We shall start with a discussion of the "restricted three-body problem"; here two bodies of finite mass revolve around one another in circular orbits, and a third body of infinitesimal mass moves in their field; this situation is approximately realized in many instances in the solar system.

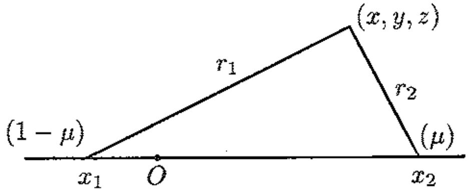

Figure 8.1

Let the origin be at the center of mass of the two finite masses and take axes rotating with the masses, such that they lie along the x-axis. Take the unit of mass to be the sum of their masses, and let the separate masses be μ and $(1-\mu)$, where $\mu \leq \frac{1}{2}$. The axes will be rotating with constant angular velocity, ω, say, and the bodies will be fixed at $(x_2, 0, 0)$ and $(x_1, 0, 0)$, where x_1 is negative. (See Figure 8.1.) Let the unit of distance be $(-x_1 + x_2)$, and let the unit of time be such as to make $k = 1$. Then, in these units,

$$\begin{aligned}\omega &= k\sqrt{\frac{(1-\mu)+\mu}{(-x_1+x_2)^3}}\\ &= 1.\end{aligned}$$

Let the infinitesimal mass be at (x, y, z) and let

$$(x - x_1)^2 + y^2 + z^2 = r_1^2$$

and

$$(x - x_2)^2 + y^2 + z^2 = r_2^2.$$

If v is the speed of the infinitesimal mass with respect to the moving axes,

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

and the modified energy integral (see (3.4.8)) is

$$v^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C, \tag{8.1.1}$$

where C is a constant. (8.1.1) is *Jacobi's integral.*

Five more integrals are needed to complete the solution (angular momentum obviously tells us nothing here), but these are not known. However, many properties of the motion can be found from a discussion of (8.1.1), and the following few sections will be devoted to this.

8.2 Tisserand's Criterion for the Identification of Comets

Let the infinitesimal mass have position vector $\mathbf{r}'$ with respect to nonrotating axes, with the same origin as before; then if $\hat{\mathbf{z}}$ is the axis of rotation,

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}'}{dt} - \hat{\mathbf{z}} \times \boldsymbol{\rho},$$

where $\boldsymbol{\rho}$ is the vector with components $(x', y', 0)$ or $(x, y, 0)$. Then

$$\begin{aligned} \left(\frac{d\mathbf{r}}{dt}\right)^2 &= \left(\frac{d\mathbf{r}'}{dt}\right)^2 - 2\left(\frac{d\mathbf{r}'}{dt}\right)\cdot(\hat{\mathbf{z}} \times \boldsymbol{\rho}) + \rho^2 \\ &= \left(\frac{d\mathbf{r}'}{dt}\right)^2 - 2\hat{\mathbf{z}}\cdot\left(\boldsymbol{\rho} \times \frac{d\mathbf{r}'}{dt}\right) + \rho^2 \\ &= \left(\frac{d\mathbf{r}'}{dt}\right)^2 - 2\hat{\mathbf{z}}\cdot\left(\mathbf{r}' \times \frac{d\mathbf{r}'}{dt}\right) + x^2 + y^2. \end{aligned}$$

Jacobi's integral becomes

$$\dot{\mathbf{r}}'^2 - 2\hat{\mathbf{z}}\cdot(\mathbf{r}' \times \dot{\mathbf{r}}') = \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C. \tag{8.2.1}$$

Now let the infinitesimal mass be a periodic comet, and $(1 - \mu)$ and μ the masses of the Sun and Jupiter, respectively, so that $\mu \sim 10^{-3}$. If we find the position and velocity of the comet at any time, from observations, we shall calculate elements from

$$\dot{\mathbf{r}}'^2 = \frac{2}{r} - \frac{1}{a}$$

and

$$\hat{\mathbf{z}}\cdot(\mathbf{r}' \times \dot{\mathbf{r}}') = \hat{\mathbf{z}}\cdot\mathbf{h} = \sqrt{a(1-e^2)}\cos i.$$

Substituting these expressions into (8.2.1), we find

$$\frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i = \frac{2}{r} - \frac{2(1-\mu)}{r_1} - \frac{2\mu}{r_2} + C.$$

Now r is nearly equal to r_1 so that, approximately,

$$\frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i = 2\mu\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + C. \tag{8.2.2}$$

Suppose the comet to be observed before and after a close approach to Jupiter; provided the comet is far from Jupiter when observed, r_1 and r_2 will be large and nearly equal, and we have approximately

$$\frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i = C. \tag{8.2.3}$$

Now, because of perturbations by Jupiter, the elements of the comet will have changed, and it is possible that they will have changed so considerably that identification is difficult (i.e., we are not sure whether it is the old comet or a new one). But C is constant throughout, so if a_1, e_1, i_1 refer to the old orbit and a_2, e_2, i_2 to the new orbit, we must have approximately

$$\frac{1}{a_1} + 2\sqrt{a_1(1-e_1^2)}\cos i_1 = \frac{1}{a_2} + 2\sqrt{a(1-e_2^2)}\cos i_2.$$

This criterion is due to Tisserand.

8.3 The Surfaces of Zero Relative Velocity

If we put $v = 0$ in (8.1.1), we have the equation

$$x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} = C. \tag{8.3.1}$$

For some value of C this will be the locus of surfaces in space to be described shortly. If we consider (8.1.1) as a function of v^2, then we see that v^2 changes sign when a surface is crossed (as long as the crossing does not take place at a double point). Hence the motion can take place on one side of the surface but not on the other. This is similar to the theorem in the problem of two bodies, that the finite motion is restricted within a circle of radius $2a$ (also deduced from

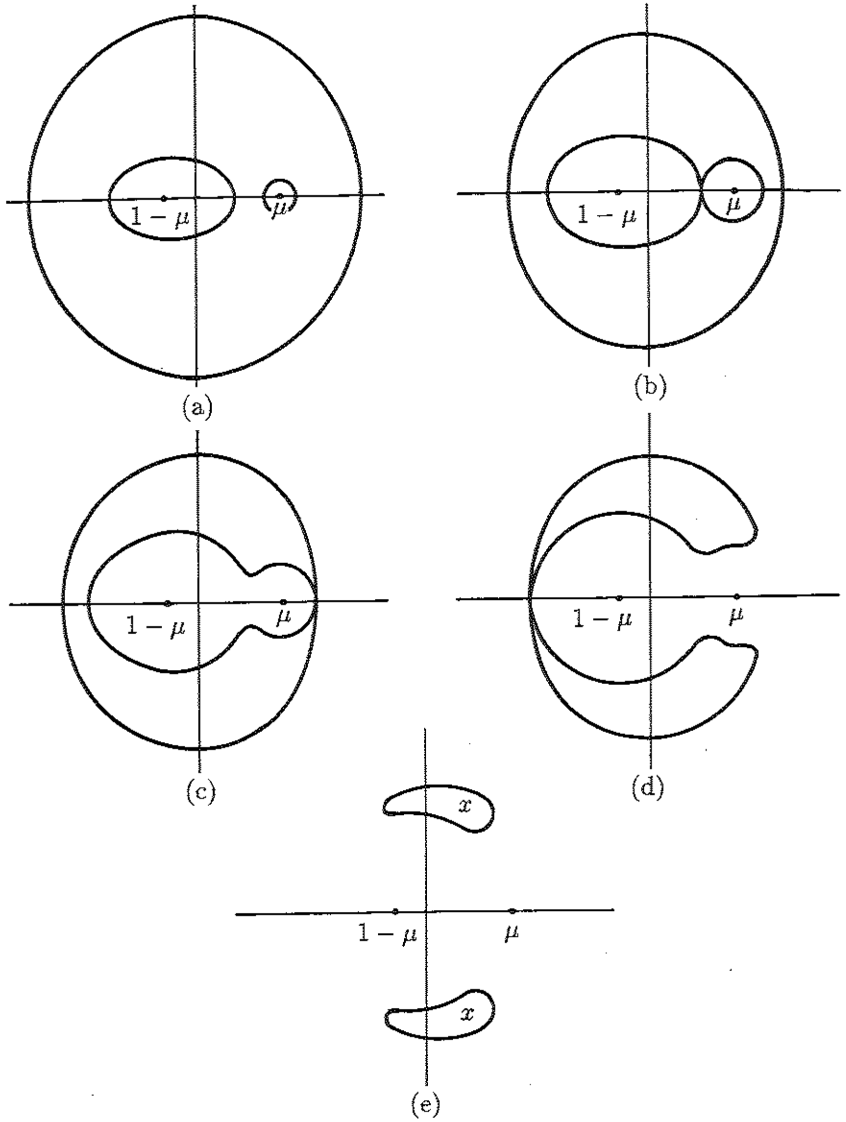

Figure 8.2

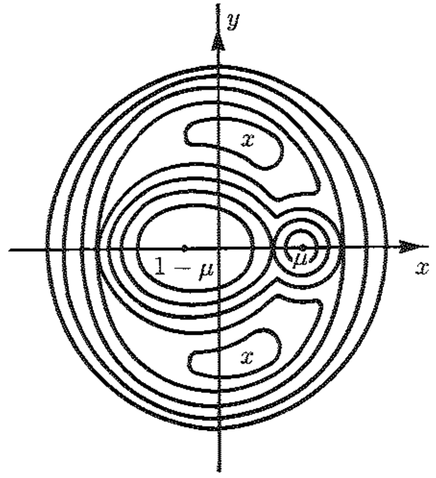

Figure 8.3

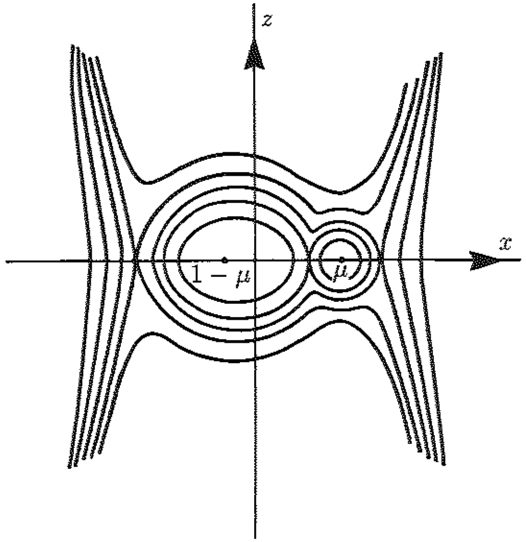

Figure 8.4

the energy integral). The construction of the surfaces is described in some detail in Moulton (Ref. 11, pp. 281-290); we shall give only a rough description here. If C is large (it must, of course, be positive), we have the three alternatives:

$$x^2 + y^2 = C \quad \text{(nearly)}$$

or

$$r_1 \text{ is very small}$$

or

$$r_2 \text{ is very small.}$$

Hence we have, roughly, a cylinder with circular cross-section, parallel to the z-axis, and two small ovoids surrounding the finite masses. The larger ovoid is around the heavier mass, $1 - \mu$. For this value of C the motion of the infinitesimal body can take place outside the cylinder or inside one of the ovoids. The situation in the x-y plane is illustrated in Figure 8.2a.

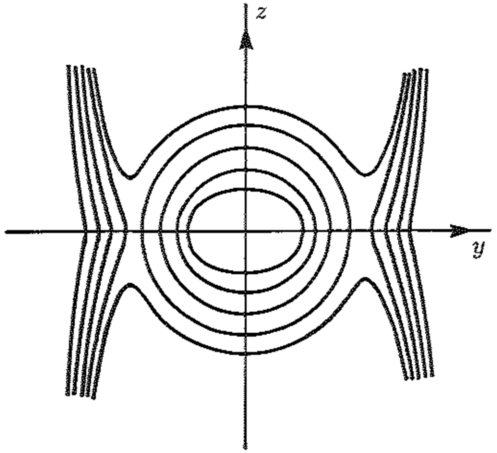

Figure 8.5

Now let C decrease. The "cylinder" shrinks (and acquires a "waist" in the x-y plane), and the ovoids expand until they coalesce; this will take place in the x-y plane at a point closer to μ than to $1 - \mu$. This is illustrated in Figure 8.2b.

As C decreases further, the wall of the cylinder meets the smaller and later the larger of the ovoids (see Figures 8.2c and 8.2d). Finally we are left with two tadpole-like shapes that eventually shrink to points (see Figure 8.2e). The parts of Figure 8.2 can be combined into one, and this is shown in Figure 8.3. Similar sections in the x-z and y-z planes are shown in Figures 8.4 and 8.5.

Figures 8.6 and 8.7 show two aspects of the surface

$$z = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2}$$

where $r_1^2 = (x+\mu)^2 + y^2$, $r_2^2 = (x - 1 + \mu)^2 + y^2$ with $\mu = 0.4$. The level curves correspond to curves of zero velocity in Figure 8.3.

These limiting surfaces were first discussed by Hill in relation to the motion of the Moon.

If you have graphics facilities on your computer, you should write a program to generate these curves. See Ref. 45 for some remarkable computer-generated representations of the three-dimensional surfaces.

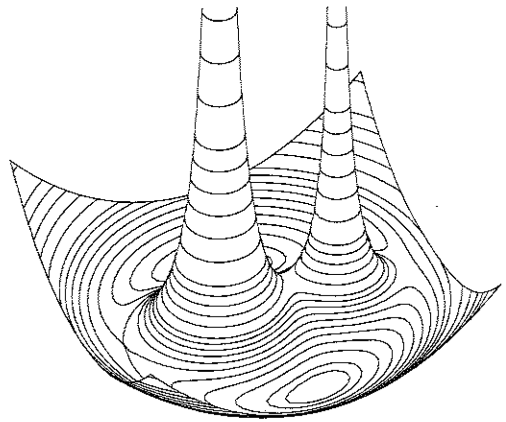

Figure 8.6

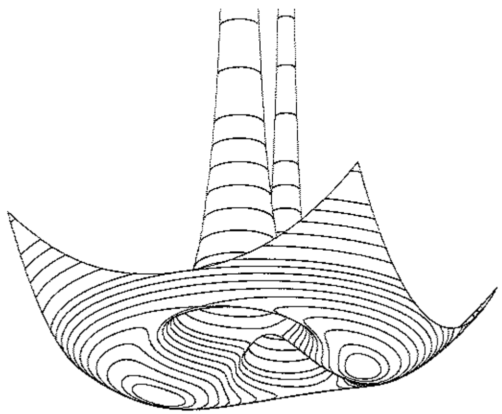

Figure 8.7

8.4 The Positions of Equilibrium

All the double points involved in the surfaces discussed in Section 8.3 occur in the x-y plane. We might expect these to have some significance in the solution, and they are, in fact, positions of equilibrium, as we shall now show.

Let V be the "modified potential"; then

$$-V = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}. \qquad (8.4.1)$$

The equation of motion is

$$\ddot{\mathbf{r}} + 2\hat{\mathbf{z}} \times \dot{\rho} = -\nabla V. \qquad (8.4.2)$$

Suppose the infinitesimal body to be initially at rest (with respect to the rotating axes); it will start to move off in the direction of $-\nabla V$. The surfaces described in Section 8.3 are given by the function

$$F(x,y,z) \equiv V + \tfrac{1}{2}C = 0. \qquad (8.4.3)$$

Now the normal at any point on a surface has direction cosines that are proportional to $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$, so it is in the direction of $-\nabla V$. Therefore, the infinitesimal body, initially at rest, will start to move off in the direction of the normal to the surface through the point which it occupies. But suppose this is to be a double point; then there is not a unique normal. The body has no reason for moving off in one direction rather than another, and unless it receives a small nudge it will stay where it is. (This is similar to the situation of the student, who, having two lectures of equal importance at the same time, goes to neither.) The double points are therefore positions of equilibrium. The double points occur at stationary values of F, the condition for this being

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0.$$

Now we have

$$\frac{\partial F}{\partial z} \equiv z\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right)$$

so that any double point occurs in the x-y plane, for which $z = 0$. Then we can put $z = 0$, and find the conditions for $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$. They are

$$x - (1-\mu)\frac{x-x_1}{r_1^3} - \mu\frac{x-x_2}{r_2^3} = 0 \qquad (8.4.4)$$

and

$$y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} = 0. \qquad (8.4.5)$$

(8.4.5) is satisfied if $y = 0$, and in this case (8.4.4) becomes

$$x - (1-\mu)\frac{x-x_1}{|x-x_1|^3} - \mu\frac{x-x_2}{|x-x_2|^3} = 0. \qquad (8.4.6)$$

(Here $|a|$ indicates that the positive value of a is used.) The left-hand side of (8.4.6), considered as a function of x, has a graph of the form shown in Figure 8.8. The verification of this graph (after the fashion of that of Figure 5.6) is left as an exercise to the reader. There are, then, three real roots, and this was to be expected from the three double points on the x-axis appearing in Figure 8.3. These are called the *Lagrangian points*, L_1, L_2, and L_3.

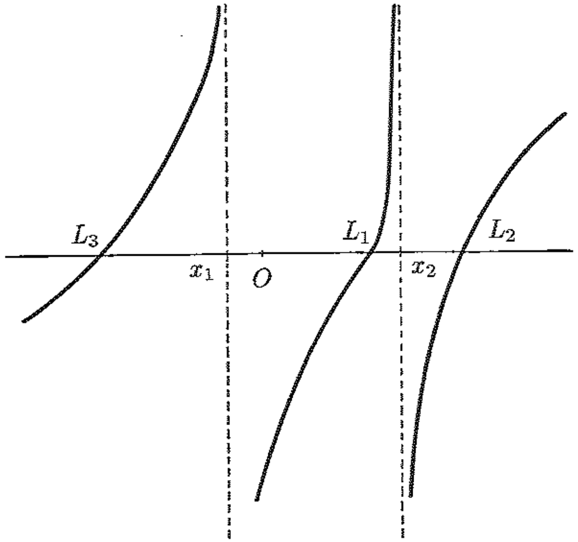

Figure 8.8

Now consider what happens when $y \neq 0$. (8.4.5) becomes

$$1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} = 0.$$

Multiply this by $(x - x_2)$ and $(x - x_1)$ and subtract the products separately from (8.4.4). We find

$$x_2 - (1-\mu)\frac{x_2 - x_1}{r_1^3} = 0$$

and

$$x_1 - \mu\frac{x_1 - x_2}{r_1^3} = 0.$$

But $x_2 = 1 - \mu$ and $x_1 = -\mu$, since the origin is at the center of mass of the finite masses, so that these equations reduce to

$$r_1 = r_2 = 1.$$

These points of equilibrium form equilateral triangles in the x-y plane with the two finite masses. They are the *Lagrangian points* L_4 (x positive) and L_5.

8.5 The Stability of the Points of Equilibrium

In the preceding section we argued that the infinitesimal body would remain at a double point unless it received a slight nudge. If, for *any* such nudge, the motion that follows consists of small oscillations about the double point, the position of equilibrium is said to be *stable*; if, for *some* possible nudges, the body recedes indefinitely from the double point, the position is *unstable*. Therefore, to find out whether a position of equilibrium found above is stable, we must investigate the motion of the infinitesimal body if it is slightly displaced from that position. In practice (unless we assume the three bodies to be totally isolated) this will be constantly happening as a result of perturbations.

The word "stability" has many interpretations. Here, we consider "linear stability."

Let the position of equilibrium be at (x, y, z) and let the body be displaced to $(x+\xi, y+\eta, z+\zeta)$ where ξ, η, and ζ are small; we assume, until it may be proved to the contrary, that these quantities remain small. If we neglect their squares and products and remember that $\partial V/\partial x = 0$, etc., at (x, y, z), then the equations of motion are

$$\left.\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= -\xi V_{xx} - \eta V_{xy} - \zeta V_{xz}, \\ \ddot{\eta} + 2\dot{\xi} &= -\xi V_{yx} - \eta V_{yy} - \zeta V_{yz}, \\ \ddot{\zeta} \qquad &= -\xi V_{zx} - \eta V_{zy} - \zeta V_{zz}, \end{aligned}\right] \tag{8.5.1}$$

where V_{yz} stands for $\partial^2 V/\partial y \partial z$, etc., and these quantities are evaluated at the point of equilibrium so that they have constant values.

Now

$$\begin{aligned} -V = \tfrac{1}{2}(x^2+y^2) &+ (1-\mu)\{(x-x_1)^2+y^2+z^2\}^{-1/2} \\ &+ \mu\{(x-x_2)^2+y^2+z^2\}^{-1/2}. \end{aligned}$$

Define α and β by

$$\alpha = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}$$

and

$$\beta = \frac{1-\mu}{r_1^5} + \frac{\mu}{r_2^5};$$

then we find

$$\begin{aligned} V_{xx} &= -1 + \alpha - 3(1-\mu)\frac{(x-x_1)^2}{r_1^5} - 3\mu\frac{(x-x_2)^2}{r_2^5}, \\ V_{yy} &= -1 + \alpha - 3y^2\beta, \\ V_{zz} &= \alpha - 3z^2\beta, \\ V_{yz} &= -3yz\beta, \end{aligned}$$

$$\begin{aligned} V_{zx} &= -3(1-\mu)\frac{z(x-x_1)}{r_1^5} - 3\mu\frac{z(x-x_2)}{r_2^5}, \\ V_{xy} &= -3(1-\mu)\frac{y(x-x_1)}{r_1^5} - 3\mu\frac{y(x-x_2)}{r_2^5}. \end{aligned}$$

For a straight-line solution for which $y = z = 0$ and $x = x_0$, say, we have

$$r_1^2 = (x_0 - x_1)^2 \quad \text{and} \quad r_2^2 = (x_0 - x_2)^2.$$

Also

$$V_{yz} = V_{zx} = V_{xy} = 0,$$

so the equations of motion following a small displacement are

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi(1+2\alpha), \\ \ddot{\eta} + 2\dot{\xi} &= \eta(1-\alpha), \\ \ddot{\zeta} &= -\zeta\alpha. \end{aligned}$$

Since α is positive, the oscillations in the z-direction are finite and small, and therefore stable, and we can concentrate on the first two equations.

We try a solution $\xi = Ke^{\lambda t}$, $\eta = Le^{\lambda t}$. If λ is purely imaginary, then the solution can be put entirely in the form of sines and cosines, and the motion, consisting of finite oscillations, will be stable. Otherwise the solution must involve hyperbolic functions; then ξ and η will increase without limit, and the motion is unstable. Substituting the trial solutions into the equations of motion, we have

$$K\lambda^2 - 2L\lambda = K(1+2\alpha)$$

and

$$L\lambda^2 + 2K\lambda = L(1-\alpha).$$

Eliminating K and L, we find

$$\begin{vmatrix} \lambda^2 - (1+2\alpha) & -2\lambda \\ 2\lambda & \lambda^2 - (1-\alpha) \end{vmatrix} = 0$$

or

$$\lambda^4 + \lambda^2(2-\alpha) + (1+2\alpha)(1-\alpha) = 0.$$

For stability λ must be purely imaginary so that there must be two real negative roots for λ^2; for this to be so, we must at least have

$$1 - \alpha > 0,$$

since the product of the roots must be positive, or

$$1 - \frac{1}{r_1^3} + \mu\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) > 0.$$

From Figure 8.6 we see that (8.4.6) has:

- a positive root greater than x_2, when r_1 is greater than r_2;
- a positive root between x_1 and x_2, when r_1 is greater than r_2;
- a negative root less than x_1, when r_1 is less than r_2.

Now (8.4.6) can be written in the form

$$x_0(1-\alpha)+(1-\mu)\frac{x_1}{r_1^3}+\mu\frac{x_2}{r_2^3}=0$$

or, since $x_1=-\mu$ and $x_2=1-\mu$,

$$x_0(1-\alpha)-\mu(1-\mu)\left\{\frac{1}{r_1^3}-\frac{1}{r_2^3}\right\}=0.$$

So

$$(1-\alpha)=\frac{\mu(1-\mu)}{x_0}\left\{\frac{1}{r_1^3}-\frac{1}{r_2^3}\right\}.$$

It is clear from inspection that for each solution $(1-\alpha)$ is negative. Therefore the straight-line solutions are unstable. Actually some displacements lead to finite oscillations (such as the oscillations in the z-direction), but for stability all possible displacements must lead to these. When testing for stability, it is not sufficient to try some special displacements.

If we assume that the orbit of the Earth around the Sun is circular, we can find the three straight-line positions of equilibrium, one of which is on the side of the Earth away from the Sun. It has been argued that meteoric material may be temporarily trapped in this position and that this explains a very faint glow seen in the night sky in a position exactly opposite the (invisible) Sun, called the *gegenschein*; the glow would be the reflection of sunlight from the trapped material. A meteoric particle trapped in this position must remain there for a short time if it were lucky in its perturbations, but it would eventually move away. The reader may object to the statement made earlier that the particle would "recede indefinitely" from the position of equilibrium; the objection is valid, since in this example the displaced meteoric material will remain indefinitely within the solar system. But the phrase is intended to apply only to the perturbed equations (8.5.1), which are approximate, and not necessarily to motion farther afield.

Next consider the stability of the triangular solutions. We have $r_1=r_2=1$, so that

$$x=\frac{1}{2}(1-2\mu),\quad y=\pm\frac{\sqrt{3}}{2},\quad \textit{and}\quad z=0.$$

We can take the positive sign for y without loss of generality. The values of V_{ij} that are not zero are

$$V_{xx}=-\frac{3}{4},$$

$$V_{yy}=-\frac{9}{4},$$
$$V_{zz}=1,$$
$$V_{xy}=-\frac{3\sqrt{3}}{4}(1-2\mu).$$

The equations of motion are

$$\ddot{\xi}-2\dot{\eta}=\frac{3}{4}\xi+\frac{3\sqrt{3}}{4}(1-2\mu)\eta,$$
$$\ddot{\eta}+2\dot{\xi}=\frac{9}{4}\eta+\frac{3\sqrt{3}}{4}(1-2\mu)\xi,$$
$$\ddot{\zeta}=-\zeta.$$

The oscillations in the z-direction are stable, the period being the same as that of the revolution of the finite bodies. To investigate the first two equations, we again try $\xi=Ke^{\lambda t}$, $\eta=Le^{\lambda t}$. Substituting, and eliminating K and L as before, we find

$$\lambda^4+\lambda^2+\frac{27}{4}\mu(1-\mu)=0.$$

Considered as a quadratic for λ^2, this equation must have real negative roots for stability. But the sum of the roots is negative and their product is positive; so, provided the roots are real, they must be negative. Hence, to find the condition for stability, it is sufficient to find the condition for real roots. This is

$$1-4\cdot\frac{27}{4}\mu(1-\mu)>0$$

or

$$1-27\mu(1-\mu)>0.$$

Putting the left-hand side equal to ϵ, and solving the resulting quadratic for μ, we find

$$\mu=\frac{1}{2}\pm\sqrt{\frac{23+4\epsilon}{108}}.$$

Since $\mu\leq\frac{1}{2}$, we take the lower sign. The condition for stability is $\epsilon>0$, or

$$\mu<\frac{1}{2}-\sqrt{\frac{23}{108}}=0.385\ldots=\mu_1.$$

This condition is satisfied for Jupiter and the Sun, and in fact we do find some minor planets, called *the Trojans*, oscillating about the triangular positions. These oscillations are considerable, amounting to as much as 20° in longitude so that the small oscillations considered above cannot be applied to the actual motion; they merely ensure linear stability.

The investigation of general, non-linear stability requires mathematics and industry of a higher order than that applied here. The triangular points have been proved stable for $0<\mu<\mu_1$, excluding the values $\mu_2=\frac{1}{2}\left(1-\frac{1}{45}\sqrt{1833}\right)=0.0243\ldots$ and $\mu_3=\frac{1}{2}\left(1-\frac{2}{45}\sqrt{117}\right)=0.0135\ldots$. See problem 26 at the end of this chapter.

8.6 The Lagrangian Solutions for the Motion of Three Finite Bodies

The equations of motion for three finite bodies are

$$\left.\begin{aligned}\ddot{\mathbf{r}}_1 &= -m_2\frac{\mathbf{r}_1-\mathbf{r}_2}{r_{12}^3} - m_3\frac{\mathbf{r}_1-\mathbf{r}_3}{r_{13}^3},\\ \ddot{\mathbf{r}}_2 &= -m_3\frac{\mathbf{r}_2-\mathbf{r}_3}{r_{23}^3} - m_1\frac{\mathbf{r}_2-\mathbf{r}_1}{r_{21}^3},\\ \ddot{\mathbf{r}}_3 &= -m_1\frac{\mathbf{r}_3-\mathbf{r}_1}{r_{31}^3} - m_2\frac{\mathbf{r}_3-\mathbf{r}_2}{r_{32}^3}.\end{aligned}\right\} \tag{8.6.1}$$

Here $r_{ij} = |\mathbf{r}_j - \mathbf{r}_i|$; the unit of time is chosen to make $k = 1$.

If the equations are multiplied by m_1, m_2, and m_3, respectively, and added, the terms on the right-hand side vanish, and the equation can be integrated to yield the fact that the center of mass of the three bodies moves in a straight line with constant speed. We can, then, take the center of mass as a new origin, so that

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 = \mathbf{0}. \tag{8.6.2}$$

We shall look for solutions for which the geometrical form of the configuration remains the same. The scale can change and the figure can rotate, but that is all. A solution of this type is called a *homographic solution.* We have already found two such solutions for the restricted three-body problem, where the scale remains constant. We shall start by seeing if the equations (8.6.1) can be satisfied if the configuration is an equilateral triangle. In this case,

$$r_{23} = r_{31} = r_{12} = r,$$

where r need not be constant. On substitution, the first equation of (8.6.1) becomes

$$\ddot{\mathbf{r}}_1 = -\frac{1}{r^3}[m_2(\mathbf{r}_1-\mathbf{r}_2) + m_3(\mathbf{r}_1-\mathbf{r}_3)]. \tag{8.6.3}$$

From (8.6.2) we have

$$m_2(\mathbf{r}_1-\mathbf{r}_2) + m_3(\mathbf{r}_1-\mathbf{r}_3) = \mathbf{r}_1(m_1+m_2+m_3) \tag{8.6.4}$$

so that (8.6.3) becomes

$$\ddot{\mathbf{r}}_1 = -(m_1+m_2+m_3)\frac{\mathbf{r}_1}{r^3}. \tag{8.6.5}$$

Squaring (8.6.4), we find

$$\begin{aligned}\mathbf{r}_1^2(m_1+m_2+m_3)^2 &= m_2^2r_{12}^2 + 2m_2m_3r_{12}r_{13}\cos 60^\circ + m_3^2r_{13}^2\\ &= r^2(m_2^2 + m_2m_3 + m_3^2).\end{aligned}$$

So (8.6.5) becomes

$$\ddot{\mathbf{r}}_1 = -M_1\frac{\mathbf{r}_1}{r_1^3}, \tag{8.6.6}$$

where

$$M_1 = \frac{(m_2^2 + m_2m_3 + m_3^2)^{3/2}}{(m_1+m_2+m_3)^2}.$$

Hence m_1 moves in a central orbit around the fixed center of mass as though a mass M_1 were located there; it will continue to do so as long as the configuration of the three bodies is maintained. Similar results follow for the other two bodies. Then, as long as the initial conditions are right, the equations of motion will ensure that the figure remains an equilateral triangle. Initially, we must obviously have the masses at the apices of an equilateral triangle, and the velocities must be such as to cause the figure to remain an equilateral triangle, independently of the accelerations caused by the gravitational forces. For this to be so, the initial velocities must be proportional to the distances r_i, and must make equal angles with the directions $\hat{\mathbf{r}}_i$.

We notice that in the case considered above, the resultant force $\mathbf{F}_i$, acting on m_i, passes through the center of mass, and that if F_i is force per unit mass, then

$$F_1 : F_2 : F_3 = r_1 : r_2 : r_3. \tag{8.6.7}$$

These two conditions, together with those ensuring that the initial conditions are right, are necessary and sufficient for the configuration of masses to remain in the same original geometrical form, as we shall now show.

The motion of a homographic solution is confined to a plane. I know of no concise proof that this is the case, and it will be assumed in this discussion. (For a proof that is not concise, see Ref. 44).

We start by showing that the first condition is necessary. The fact that the shape of the configuration is maintained means that the relative distances are given by

$$\frac{r_{23}}{\mathring{r}_{23}} = \frac{r_{31}}{\mathring{r}_{31}} = \frac{r_{12}}{\mathring{r}_{12}} = \lambda(t),$$

where the zero superscript indicates the value at t_0. (8.6.4) is true generally; squaring both sides we find

$$\begin{aligned}(m_1+m_2+m_3)^2r_1^2 &= m_2^2r_{12}^2 + m_3^2r_{13}^2 + 2m_2m_3r_{12}r_{13}\cos\theta_{23}\\ &= [\lambda(t)]^2(m_2^2\mathring{r}_{12}^2 + m_3^2\mathring{r}_{13}^2 + 2m_2m_3\mathring{r}_{12}\mathring{r}_{13}\cos\theta_{23}).\end{aligned}$$

Here θ_{23} is the angle between $(\mathbf{r}_1 - \mathbf{r}_2)$ and $(\mathbf{r}_1 - \mathbf{r}_3)$; since this angle is constant we deduce

$$r_1 = \mathring{r}_1\lambda(t),$$

and, generally,

$$r_i = \mathring{r}_i\lambda(t).$$

It follows that the shape of a triangle formed by the origin and any two masses is constant, and therefore that the angle subtended at the origin by the line joining any two masses is constant. Then if $\dot{\theta}_i$ is the angular velocity of m_i about the origin, we must have

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}(t).$$

But the total angular momentum of the system about the origin is constant and equal to

$$(m_1\mathring{r}_1^2 + m_2\mathring{r}_2^2 + m_3\mathring{r}_3^2)\lambda^2\dot{\theta}.$$

Hence the angular momentum of each individual mass about the origin is constant, so that the net force acting on that mass passes through the origin.

The equation of motion for m_i can be written

$$\begin{aligned} m_iF_i &= m_i(\ddot{r}_i - r_i\dot{\theta}_i^2) \\ &= r_im_i\left(\frac{\ddot{\lambda}}{\lambda} - \dot{\theta}^2\right). \end{aligned}$$

Hence we have the condition (8.6.7), which must, therefore, be necessary.

To show that the conditions are sufficient, we simply follow up their consequences. We have $\mathbf{r}_1 \times \mathbf{F}_1 = \mathbf{0}$, or $\mathbf{r}_1 \times \ddot{\mathbf{r}}_1 = \mathbf{0}$. Applying this condition to the first equation of (8.6.1) we find

$$\mathbf{r}_1 \times \left\{\frac{m_2\mathbf{r}_2}{r_{12}^3} + \frac{m_3\mathbf{r}_3}{r_{13}^3}\right\} = \mathbf{0}$$

or, using (8.6.2),

$$m_2\mathbf{r}_1 \times \mathbf{r}_2\left\{\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3}\right\} = \mathbf{0},$$

with similar results for the other masses. For these equations to be satisfied, we must either have

$$r_{23} = r_{31} = r_{12},$$

the case that we have already investigated, or

$$\mathbf{r}_2 \times \mathbf{r}_3 = \mathbf{r}_3 \times \mathbf{r}_1 = \mathbf{r}_1 \times \mathbf{r}_2 = \mathbf{0},$$

so that the particles lie in a straight line; this satisfies our geometrical conditions. There are no other possibilities.

We might have suspected the possibility of straight-line solutions by analogy with the restricted three-body problem; now we have to justify them for the motion of three finite bodies and find the number of possible solutions. Let the line be the x-axis, and let it rotate with angular velocity $\dot{\theta}$. We have $x_i = \mathring{x}_i\lambda(t)$. The force acting on m_1 is

$$\begin{aligned} F_1 &= -m_2\frac{x_1 - x_2}{x_{12}^3} - m_3\frac{x_1 - x_2}{x_{13}^3} \\ &= -\frac{1}{\lambda^2}\left\{m_2\frac{\mathring{x}_1 - \mathring{x}_2}{\mathring{x}_{12}^3} - m_3\frac{\mathring{x}_1 - \mathring{x}_3}{\mathring{x}_{13}^3}\right\}. \end{aligned}$$

Now λ is proportional to the distance, so that m_1 is effectively acted upon by a force directed toward the center of mass and obeying the inverse square law; it will move in a conic orbit and so will the other two masses.

To show that this solution is possible, we must impose the condition

$$F_1 : F_2 : F_3 = x_1 : x_2 : x_3.$$

If this is true initially, it will always be true, so it is sufficient to prove that for the initial conditions,

$$Rx_1 = m_2\frac{x_1 - x_2}{x_{12}^3} + m_3\frac{x_1 - x_3}{x_{13}^3}, \tag{8.6.8}$$

$$Rx_2 = m_3\frac{x_2 - x_3}{x_{23}^3} + m_1\frac{x_2 - x_1}{x_{21}^3}, \tag{8.6.9}$$

$$Rx_3 = m_1\frac{x_3 - x_1}{x_{31}^3} + m_2\frac{x_3 - x_2}{x_{32}^3}. \tag{8.6.10}$$

Here R is a constant that depends on the initial conditions.

Let us prescribe a scale such that the initial distance $x_1 - x_2 = 1$. We need to show that the third body can be given some value of x such that the initial conditions are satisfied. Let

$$x_2 - x_3 = X.$$

We shall investigate whether there is an x_3 such that x_2 lies between x_1 and x_3: in this case X must be positive. Subtracting (8.6.9) from (8.6.8), and (8.6.10) from (8.6.9), we find

$$\left.\begin{aligned} R &= m_1 + m_2 - \frac{m_3}{X^2} + \frac{m_3}{(1+X)^2}, \\ &\text{and} \\ RX &= \frac{m_3}{X^2} - m_1 + \frac{m_1}{(1+X)^2} + \frac{m_2}{X^2}. \end{aligned}\right] \tag{8.6.11}$$

If we eliminate R, remove fractions, and arrange in powers of X, we obtain the quintic

$$\begin{aligned} &(m_1 + m_2)X^5 + (3m_1 + 2m_2)X^4 + (3m_1 + m_2)X^3 \\ &\qquad -(m_2 + 3m_3)X^2 - (2m_2 + 3m_3)X - (m_2 + m_3) = 0. \end{aligned} \tag{8.6.12}$$

The coefficients of the powers of X change sign only once, so that by Descartes' rule of signs there is one positive root and one only. Hence there is one and only one position of m_3 such that $x_1 > x_2 > x_3$. If we want x_3 to lie between x_1 and x_2, equations (8.6.11) are slightly different, but we again finish with a quintic that has only one relevant root. The same applies if we want x_1 to lie between x_2 and x_3. Accordingly, there are three possible straight-line configurations and three only; these correspond to those found for the restricted three-body problem.

This completes the discussion of these solutions, called the *Lagrangian solutions*, to the three-body problem. The equilateral triangle solutions and the straight-line solutions are the only ones satisfying the geometrical conditions that there can only be rotation and change of scale. If there is no change of scale, the solutions are called *stationary*, and the positions of the bodies are called *libration points*. Thus the Trojan planets oscillate about libration points, and the gegenschein (if the suggested explanation is correct) is at a libration point, the other bodies being the Sun and Jupiter, and the Sun and the Earth, respectively.

8.7 Problems

1. Amend the criterion for the recognition of comets to apply to parabolic orbits. Show that the perturbations by Jupiter on a parabolic comet moving in the plane of Jupiter's orbit cannot result in the orbit's remaining parabolic, and find the relation between a and e of the resulting orbit. Consider some particular cases.

2. Show that if a parabolic comet is perturbed by Jupiter into another parabolic orbit, the relation between the old and new inclinations is given by
$$\cos i_1 = \sqrt{2} \cos i_2.$$

3. Show analytically that the double point in the restricted three-body problem, which lies between the two finite masses, is closer to the lighter of the masses, and also that it is closer to μ than the double point lying on the far side.

4. Investigate the straight-line solution on the far side of the lighter mass when μ is small. Show that if $x = 1 - \mu + \rho$, then ρ can be expanded in a power series in $\mu^{1/3}$, in which the first three terms are
$$\rho = \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} - \frac{1}{9}\left(\frac{\mu}{3}\right)^{3/3}.$$

5. Find series, corresponding to that in problem 4, for the other straight-line solutions.

6. Assuming that the suggestion made in the text about the gegenschein is correct, verify that the position of the libration point is not such that the meteors will lie in the Earth's shadow.

7. Consider motion in the x-y plane in the restricted three-body problem. Show that the equation of a limiting curve can be written
$$(1-\mu)\left(r_1^2 + \frac{2}{r_1}\right) + \mu\left(r_2^2 + \frac{2}{r_2}\right) = C + \mu(1-\mu),$$
and hence show that the minimum value of C is $3 - \mu(1-\mu)$.

8. Show that, to the first order of small quantities, periodic orbits exist in the x-y plane about the straight-line points of libration in the restricted three-body problem. Also show that these are ellipses with major axes parallel to the y-axis, and are such that the eccentricity is independent of the initial displacement, but depends only on the distribution of the masses.

9. If $(1-\mu)$ and μ are the Sun and the Earth, respectively, prove that the period of oscillation parallel to the z-axis for an infinitesimal body slightly displaced from the point of libration opposite the Sun is 183.304 mean solar days.

10. For the situation of problem 9, show that the period of small oscillations in the x-y plane is 177.0 days.

11. Show that for small values of μ, the periods of oscillation both parallel to the z-axis and in the x-y plane are, in general, longest for the point opposite to μ with respect to $(1-\mu)$ as origin; next longest for the point opposite to $(1-\mu)$ with respect to μ as origin; and shortest for the point between $(1-\mu)$ and μ.

12. Three bodies move in accordance with a Lagrangian solution. Show that the orbit of any one about any other, taken as origin, is a conic. Suppose that the masses are equal and that they form an equilateral triangle at any time. Let P be the period in which they revolve around their center of mass and a be the semi-major axis of the orbit of any one body taken with respect to any other. Show that the radius of the circle in which a particle would revolve around one of the bodies in the time P is $a \cdot 3^{-1/3}$.

13. Show that the equilateral triangle circular solutions hold for the law of force μ/r^n. Investigate the general equilateral triangle solutions for this law, and also consider more complicated laws of central force.

14. Find the number of colinear solutions for the law of force μ/r^n.

15. Investigate the colinear solutions when the law of force is μ/r^3.

16. Show that when the law of force varies inversely with the fifth power of the distance, one solution is that each of the bodies moves in a circle through their center of mass in such a way that the three bodies are always at the vertices of an equilateral triangle.

17. Show that if the three bodies are placed at rest in any one of the configurations admitting circular solutions, they will fall to the center of mass in the same time in straight lines.

18. Find the distribution of mass among the three bodies for which the time of falling to their center of mass (see problem 17) will be (a) the least and (b) the greatest.

19. Solve completely the three-body problem when the law of force is proportional to the distance.

20. Investigate the possibility of a solution to the four-body problem when the masses are at the vertices of a regular tetrahedron.

21. Consider Jacobi's integral when applied approximately to the Sun, the Earth, and the (massless) Moon. Find the value of C, and investigate the size and shape of the Hill limiting surface for the motion of the Moon.

22. Consider small oscillations about the libration points for the Trojan planets. Show that of the two periods one is nearly the same as that of Jupiter, and the other is nearly 148 years. Considering the oscillation of longer period (usually known as *the libration*), show that the approximate ratio of the amplitudes along and perpendicular to the radius vector is $3\sqrt{\mu} : 1$, or $1 : 18.7$.

23. If we define

$$\Phi(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu),$$

show that Jacobi's integral can be written in the form

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Phi - C.$$

Show that C takes the same value at L_4 and L_5, independent of μ, and find this value. Show also that

$$2\Phi = (1-\mu)\left(r_1^2 + \frac{2}{r_1}\right) + \mu\left(r_2^2 + \frac{2}{r_2}\right).$$

24. Write a program to find the values of C corresponding to L_1, L_2, and L_3 for any given value of μ.

25. A spacecraft is in a circular orbit around the Earth, with altitude 100 km. It is to be given an additional tangential velocity, and the consequent orbit is to take it to the Moon. Find the minimum velocity for which this might be possible on the basis of Jacobi's integral. (Assume the orbit of the Moon to be circular. You need to find the energy corresponding to L_1.)

26. For $\mu < 0.0385\ldots$ there are oscillations around L_4, according to the linear theory, with two different frequencies, say ω_1 and ω_2. Find the values of μ for which $\omega_1 = 2\omega_2, and\ \omega_1 = 3\omega_2$. For these values of μ, it has been proved by *non-linear* analysis that L_4 and L_5 are unstable.

Chapter 9

The n-Body Problem

9.1 The Center of Mass and the Invariable Plane

In this chapter we shall examine formally the equations of the n-body problem and their known solutions, and put them into a form that is readily useful in work on perturbations.

Let a system of n bodies consist of point masses m_i at $\mathbf{r}_i$, where $i = 1, 2, \ldots, n$, and the $\mathbf{r}_i$ are expressed with respect to an inertial frame of reference. Let

$$r_{ij} = |\mathbf{r}_j - \mathbf{r}_i|, \quad (r_{ij} = r_{ji}),$$

then the equation of motion of m_i is

$$m_i\ddot{\mathbf{r}}_i = -k^2 m_i \sum_{j=1}^{n} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}^3}. \tag{9.1.1}$$

Here the summation excludes $j = i$, and this case will automatically be excluded from future summations in this chapter where it would result in the vanishing of a denominator. For a complete solution of the n-body problem, $6n$ constants of integration are needed; actually only ten are known.

If all the equations of the form (9.1.1) are added, all the terms on the right-hand side cancel out, and we have

$$\sum_{i=1}^{n} m_i\ddot{\mathbf{r}}_i = \mathbf{0}.$$

This can be integrated at once to give

$$\sum_{i=1}^{n} m_i\mathbf{r}_i = \mathbf{a}t + \mathbf{b}, \tag{9.1.2}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. This means that the center of mass of the system moves, with respect to the (inertial) system of reference, in a straight