As a clock, the Earth does not keep accurate time. Consider the prediction from theory of the position of some celestial object at some definite time. If observation does not confirm the prediction, it is possible that either the theory or the recorded time of observation may be in error. Toward the end of the nineteenth century it had become increasingly probable that observed deviations between observations and the gravitational theory of the Moon's motion were not caused by imperfections in the theory but by irregularities in the Earth's rate of rotation. Proof was lacking until, during the first half of the twentieth century, it was shown conclusively that the differences between observation and theory in the mean longitudes of Mercury, Venus, and the Sun exhibit fluctuations that are identical, if expressed in seconds of time, to those in the Moon's mean longitude. Over the past three centuries these fluctuations have ranged between ± 30 seconds of time. A much smaller annual variation of around one-tenth of a second in the time given by the Earth's rotation has been established with the aid of terrestrial clocks, notably the quartz crystal clock and atomic clocks. In addition ancient observations of eclipses have shown that the day is gradually becoming longer (although only by about one-thousandth of a second per day per century).

To cope with this situation, *ephemeris time* has been introduced. This runs on uniformly with an invariable basic unit, and so corresponds with the theoretical notion of time used in mechanics. It is this time that should be used in celestial mechanics. The difference between ephemeris time and universal time is tabulated in the standard almanacs.

Chapter 2

Introduction to Vectors

2.1 Scalars and Vectors

A quantity that has magnitude only is called a *scalar*; it can be represented by a number with an associated sign. For example, distance and electric charge are scalars.

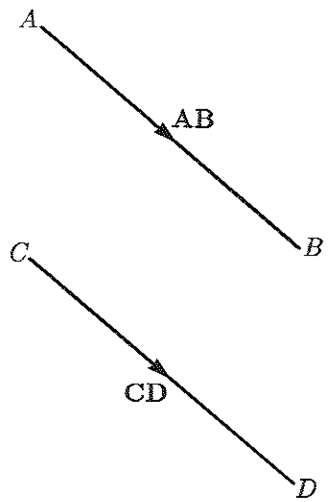

Figure 2.1

Consider two points A and B. The distance between them is AB, which is a scalar; if this distance is associated with the direction of A to B, it becomes a *vector*, $\mathbf{AB}$. A vector can be thought of as a scalar with an associated direction (this is necessary but not sufficient; vectors must also satisfy the law of addition given in the following section); or, in this example, $\mathbf{AB}$ can be regarded as a displacement of amount AB in the direction A to B.

Notation for vectors varies widely. I prefer to use an arrow, e.g., $\overrightarrow{AB}$. In this text a vector will be denoted by a symbol in boldface type. The *modulus*, or magnitude of a vector $\mathbf{a}$ is its scalar value and this will be written as $|\mathbf{a}|$, or simply a; it is always positive. A *unit vector* has unit modulus; it will be written with a "cap" above it, e.g., $\hat{\mathbf{a}}$. $\hat{\mathbf{a}}$ is the unit vector along $\mathbf{a}$; it may be referred to as the *direction* of $\mathbf{a}$ or as the *line of action* of $\mathbf{a}$. A vector with zero modulus is a *null* or *zero vector* and is simply written "0". Considering $\mathbf{AB}$ as a displacement from A to B, we have

$$\mathbf{AB} + \mathbf{BA} = \mathbf{0} \quad \text{or} \quad \mathbf{AB} = -\mathbf{BA}.$$

Vectors may be used in any of three senses:

1. *Free vectors.* Two free vectors, $\mathbf{AB}$ and $\mathbf{CD}$, are equal if they have the same modulus and parallel directions. (See Figure 2.1.)
2. *Localized vectors.* These have their lines of action passing through some point. Examples are the force acting at a point and the velocity of a point.
3. *Position vectors.* Let O be any fixed origin and P a point that may vary.

The vector

$$\mathbf{OP} = \mathbf{r}$$

is the position vector of P with respect to O. The symbol $\mathbf{r}$ will usually denote a position vector; it is always anchored to some origin. (See Figure 2.2.)

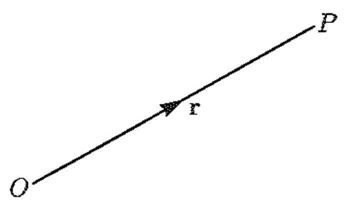

Figure 2.2

The possibility of three different senses may seem confusing, but in practice the context makes it quite clear which one is relevant.

From the definition, a vector may be multiplied by a scalar, only its modulus being affected (unless the scalar is negative, when the direction is reversed). For example, $2\mathbf{AB}$ has the same direction as $\mathbf{AB}$ but twice its modulus. So we can write

$$\mathbf{r} = r\hat{\mathbf{r}},$$

an expression which will be used rather frequently, and

$$k\mathbf{r} = (kr)\hat{\mathbf{r}}.$$

It follows from the known properties of scalars that

$$k\mathbf{r} = \mathbf{r}k$$

and

$$(k + l)\mathbf{r} = k\mathbf{r} + l\mathbf{r}.$$

2.2 The Law of Addition

Considering $\mathbf{AB}$ as a displacement, we see that two vectors $\mathbf{AB}$ and $\mathbf{BC}$ can be added to give $\mathbf{AC}$. Any two free vectors can be added as shown in Figure 2.3.

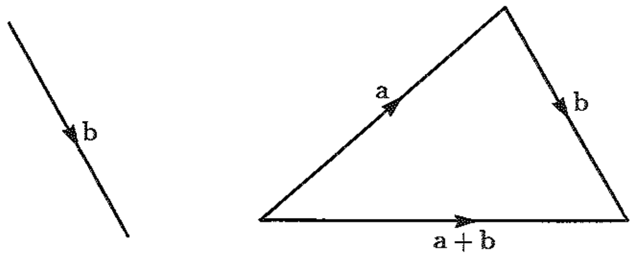

Figure 2.3

It follows from geometrical constructions (which are left to the reader) that

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad \text{(commutative law)}$$

and

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \quad \text{(associative law)}.$$

Subtraction can be considered in a similar way to addition, since $\mathbf{AB} = -\mathbf{BA}$.

To illustrate the use of vector addition, consider the straight line AD with direction $\hat{\mathbf{i}}$. Let B and C be any two points on AD, and let their position vectors with respect to any origin be $\mathbf{r}_0$ and $\mathbf{r}$. Then

$$\mathbf{OC} = \mathbf{OB} + \mathbf{BC}$$

or

$$\mathbf{r} = \mathbf{r}_0 + \lambda\hat{\mathbf{i}}, \tag{2.2.1}$$

where λ, the length BC, is a scalar. If λ varies, C traces out the whole line, so that (2.2.1) is the vector equation of the line AD. Now suppose that C is moving along the line at a constant speed v, and that at time $t = 0$, it is at B. Then at time t, $BC = \lambda = vt$, and

$$\mathbf{r} = \mathbf{r}_0 + vt\hat{\imath}. \tag{2.2.2}$$

Conversely, if equations of motion yield this solution for the motion of C, then it must be moving in a straight line with constant speed.

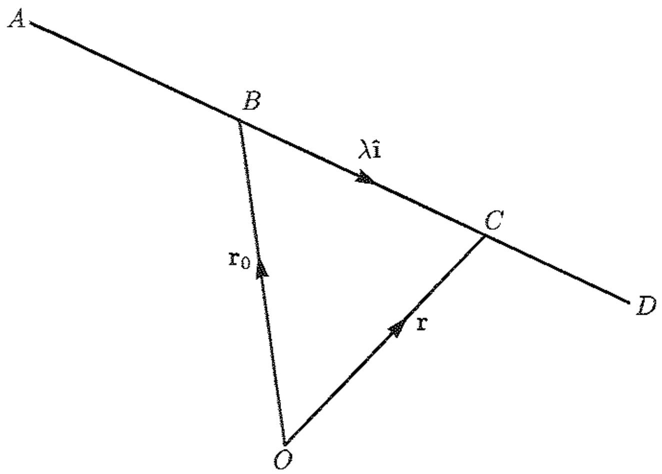

Figure 2.4

If $\hat{\imath}$, $\hat{\jmath}$, and $\mathbf{r}$ are three coplanar vectors, and $\hat{\imath}$ and $\hat{\jmath}$ are neither parallel nor antiparallel (pointing in opposite directions), then unique scalars x and y exist such that

$$\mathbf{r} = x\hat{\imath} + y\hat{\jmath}.$$

(See Figure 2.5.) Similarly, if we have another vector $\hat{\mathbf{k}}$, which does not lie in this plane, and let $\mathbf{r}$ be any (three-dimensional) vector, then unique scalars x, y, and z exist such that

$$\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}.$$

$\mathbf{r}$ is said to be *resolved* along these three directions, and x, y, and z are its *components.* Resolution can be regarded as the reverse of addition.

$\hat{\imath}$, $\hat{\jmath}$, and $\hat{\mathbf{k}}$ can be any three directions, but in the applications in this text they will be mutually perpendicular, forming a *right-handed triad.* This is such that if $\hat{\mathbf{k}}$ points away, then a rotation through 90° from $\hat{\imath}$ to $\hat{\jmath}$ will be clockwise.

Let $\mathbf{r}$ have components x, y, and z with respect to some triad; then it may be written without ambiguity as

$$\mathbf{r} = (x, y, z).$$

If another triad is chosen $\mathbf{r}$ will have different components. One great strength of the vector notation is that it is independent of any particular system of reference. Beware of resolving too early; once a vector has been resolved, there are three quantities to cope with instead of one. Also, if a different triad is needed later, then all the paraphernalia of a change of origin and a rotation of axes are needed to change the components.

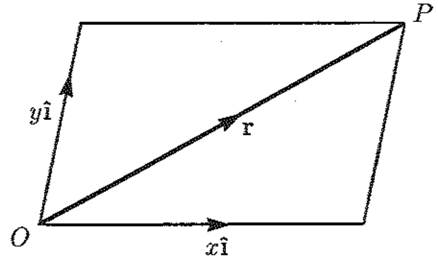

Figure 2.5a

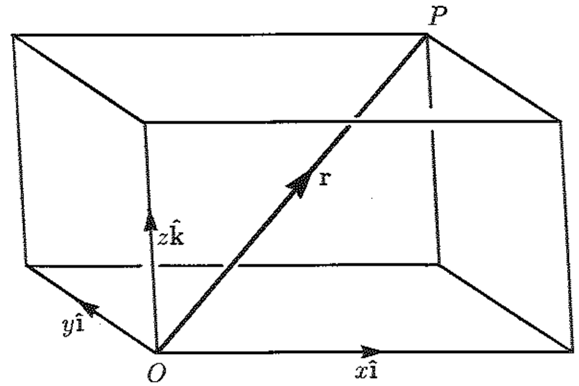

Figure 2.5b

Let $\mathbf{OP} = \mathbf{r}$ make an angle α with Ox (the x-axis). Then let

$$\cos\alpha = \frac{x}{r} = l.$$

If m and n are similarly defined, then l, m, and n are the *direction cosines* of $\mathbf{r}$ with respect to this triad. Since

$$l^2 + m^2 + n^2 = 1$$

the vector (l, m, n) has unit modulus and so is the unit vector $\hat{\mathbf{r}}$. In various notations we may write

$$\mathbf{OP} = \mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}} = (x, y, z) = (rl, rm, rn) = r(l, m, n) = r\hat{\mathbf{r}}.$$

If $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$, then

$$\mathbf{r}_1 + \mathbf{r}_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

This form of the addition law must hold for any vector. Before anything can be called a vector, it must be shown to have magnitude and direction and to obey the vector addition law. A good example of something that has magnitude and direction but is not a vector is a finite rotation. The direction is that of the axis of rotation, and the magnitude is the angle through which the rotation turns, but two finite rotations cannot be added according to the vector addition law. Infinitesimal rotations, however, can be added in this way.

Problems

1. Construct geometrically: $\mathbf{OA} + \mathbf{OB}$, $\mathbf{OA} + \mathbf{BO}$, $-\mathbf{OA} + \mathbf{BO}$.

2. State the necessary and sufficient conditions for the sum of two vectors to vanish.

3. Show by similar triangles, or otherwise, that $l(\mathbf{a} + \mathbf{b}) = l\mathbf{a} + l\mathbf{b}$.

4. Show that the resolution of a vector along any three directions (where no two are parallel or antiparallel and all three do not lie in a plane) is unique. (Consider the consequences if this were not so).

5. Find the moduli and direction cosines of the following vectors:

$$(-a, 0, 0),\quad (1, 1, -1),\quad (0, 1, -1),\quad (x, y, z)$$

and of

$$[(-a, b, 0) + (a, b, 0)],\quad [(2, 3, -5) + (3, -2, 4)]$$

and

$$[(x, y, z) + (p, q, r)].$$

All symbols may be assumed to represent positive numbers.

6. Assuming that forces are vectors, find the resultant of the two forces: P_1 along (l_1, m_1, n_1) and P_2 along (l_2, m_2, n_2).

7. Find the necessary and sufficient conditions for the sum of three nonzero vectors to vanish.

8. Show that

(a) $\dfrac{(\mathbf{OA} + \mathbf{OB})}{2}$ is the mid-point of AB;

(b) $\dfrac{(n\mathbf{OA} + m\mathbf{OB})}{(m + n)}$ is the center of mass of masses n at A and m at B;

(c) $\dfrac{(\mathbf{OA} + \mathbf{OB} + \mathbf{OC})}{3}$ is the centroid of the triangle ABC.

9. Prove by a vector method that the medians of a triangle are concurrent.

10. Two triangles ABC and $A'B'C'$ have centroids G and G'. Prove that

$$\mathbf{AA}' + \mathbf{BB}' + \mathbf{CC}' = 3\mathbf{GG}'$$

11. Express the vector equation of the line (2.2.1), in cartesians, and eliminate the parameter λ.

12. Show that the equation of a plane can be written

$$\mathbf{r} = \mathbf{r}_0 + \lambda\hat{\mathbf{a}} + \mu\hat{\mathbf{b}},$$

where $\mathbf{r}_0$ is a point in the plane, $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are any two nonparallel directions in the plane, and λ and μ are variable scalars. Show that this is equivalent to the usual cartesian equation of the plane.

13. If Ox points toward the vernal equinox, and Oz toward the celestial north pole, show that the direction cosines of a point (α, δ) are

$$(\cos\alpha\cos\delta,\quad \sin\alpha\cos\delta,\quad \sin\delta).$$

Find the direction cosines of the following points: the north pole of the ecliptic; the winter solstice; (12h, +45°); (15h, −30°); (4h 35m 53.2s, −58°16′31″). In each case verify that the sum of the squares of the direction cosines is one.

14. What is the equation of the plane of the ecliptic in the reference system of problem 13?

15. If Ox points toward the vernal equinox, and Oz toward the north pole of the ecliptic, find the direction cosines of the celestial north and south poles, the autumnal equinox, and (6h, 0°).

2.3 The Scalar Product

The angle between two vectors is conventionally taken to lie between 0 and 180°. Let the angle between $\mathbf{a}$ and $\mathbf{b}$ be θ; then their *scalar* or "dot" product is defined by

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta. \tag{2.3.1}$$

From the definition and the commutative law for scalars we have

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta = ba\cos\theta = \mathbf{b}\cdot\mathbf{a},$$

so the commutative law holds for the scalar product. $\mathbf{a}\cdot\mathbf{b}$ may be considered as the length of $\mathbf{a}$ multiplied by the projected length of $\mathbf{b}$ on $\mathbf{a}$ (i.e., $b\cos\theta$). Then, since the projection of $(\mathbf{b}+\mathbf{c})$ on $\mathbf{a}$ is the sum of the separate projections of $\mathbf{b}$ and $\mathbf{c}$ on $\mathbf{a}$, it follows that

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$$

or that the distributive law holds. We also have, from the properties of scalars,

$$m(\mathbf{a}\cdot\mathbf{b}) = (m\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(m\mathbf{b}).$$

(2.3.1) should be accepted as an arbitrary definition. Neither reason nor excuse need be offered for it, but there are plenty of reasons why it turns out to be useful in practice, and that is what matters.

One of the most important uses of the scalar product is in the resolution of vectors. In particular if $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\mathbf{k}}$ form a rectangular triad, the components of $\mathbf{r}$ in this triad are $\mathbf{r}\cdot\hat{\imath}$, $\mathbf{r}\cdot\hat{\jmath}$, and $\mathbf{r}\cdot\hat{\mathbf{k}}$. Since the directions are mutually perpendicular, we have

$$\hat{\imath}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{\imath} = 0.$$

The square of a vector, $\mathbf{r}^2$, is defined by

$$\mathbf{r}^2 = \mathbf{r}\cdot\mathbf{r} = rr = r^2.$$

In particular

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{\mathbf{k}}^2 = 1.$$

Writing $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$, we find

$$\mathbf{r}^2 = x^2 + y^2 + z^2,$$

which is the usual formula for finding the modulus of a vector.

Problems

1. If the angle between $\mathbf{AB}$ and $\mathbf{CD}$ is θ, what are the angles between $\mathbf{AB}$ and $\mathbf{DC}$ and between $\mathbf{BA}$ and $\mathbf{DC}$?

2. Find a formula for the cosine of the angle between the two vectors (x_1, y_1, z_1) and (x_2, y_2, z_2).

3. Prove by a vector method that for any triangle
$$a^2 = b^2 + c^2 - 2bc\cos A.$$

4. Solve, for the unknown vector $\mathbf{x}$, the equations:
   (a) $\mathbf{x}\cdot\mathbf{a} = 0$.
   (b) $\mathbf{x}\cdot\mathbf{a} = \mathbf{x}\cdot\mathbf{b}$ (where $\mathbf{a}\neq\mathbf{b}$).
   (c) $\mathbf{x}\cdot\mathbf{a} = b$.

5. Evaluate $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$ when $a = b$. Hence prove that the diagonals of a rhombus are perpendicular.

6. $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three nonzero and non-coplanar vectors, and no two of them are parallel or antiparallel. Show that if
$$\mathbf{x}\cdot\mathbf{a} = \mathbf{x}\cdot\mathbf{b} = \mathbf{x}\cdot\mathbf{c} = 0$$
then $\mathbf{x} = \mathbf{0}$.

7. Assuming that forces are vectors, what relation must exist between P and Q such that forces P along $(1,0,0)$ and Q along $(0,1,0)$ have zero resultant along $(l,m,0)$?

8. Show by a vector method that the altitudes of a triangle intersect in a point.

9. Show by a vector method that the perpendicular bisectors of a triangle intersect in a point.

10. Show that the equation of a plane can be written
$$(\mathbf{r}-\mathbf{r}_0)\cdot\hat{\imath} = 0.$$
Interpret $\hat{\imath}$, and show that this equation is equivalent to that of problem 12, Section 2.2.

11. Find the shortest distances from the origin to the line
$$\mathbf{r} = \mathbf{r}_0 + \lambda\hat{\imath}$$
and to the plane
$$(\mathbf{r}-\mathbf{r}_0)\cdot\hat{\imath} = 0.$$

12. Find the shortest distances from the point P_1, where $\mathbf{OP}_1 = \mathbf{r}_1$, to the line and plane of problem 11.

13. Interpret the equation
$$(\mathbf{r}-\mathbf{r}_0)^2 = a^2,$$
where $\mathbf{r}$ is a variable vector.

2.4 The Vector Product

To rotate from a vector $\mathbf{a}$ to another vector $\mathbf{b}$, the right-handed convention is adopted, so that a unique direction is associated with the rotation, as shown in Figure 2.6. Let this direction be $\hat{\imath}$; then the *vector* or "cross" product is defined by

$$\mathbf{a} \times \mathbf{b} = (ab \sin\theta)\hat{\imath}, \tag{2.4.1}$$

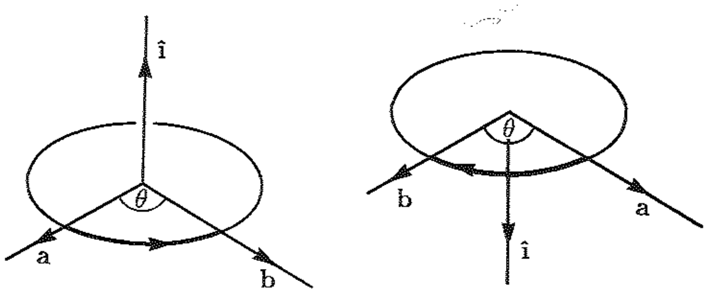

Figure 2.6

where θ is the angle between them. The order is important because, from the definition, it follows that

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

so that the vector product is not commutative. (Notation for the vector product differs, but the symbol "$\times$" seems to be nearly universal now. The author regrets the passing of the notation $\mathbf{a}\wedge\mathbf{b}$ which is much safer than $\mathbf{a} \times \mathbf{b}$, since $\times$ has so many uses; in particular it is fatally easy to start to treat $\times$ as a variable!)

The distributive law does hold; i.e.,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

There are several proofs of this, all of which require careful reasoning. The proof given here is based on that given in Rutherford's *Vector Methods* (Ref. 7).

Let a plane perpendicular to $\mathbf{a}$ be called π, and let the projections of $\mathbf{b}$ and $\mathbf{c}$ on π be $\mathbf{b}'$ and $\mathbf{c}'$. Then the projection of $(\mathbf{b}+\mathbf{c})$ on π is $(\mathbf{b}'+\mathbf{c}')$. Now the length of $\mathbf{b}'$ is $b\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$; so we have

$$\mathbf{a} \times \mathbf{b}' = \mathbf{a} \times \mathbf{b}$$

and similarly,

$$\mathbf{a} \times \mathbf{c}' = \mathbf{a} \times \mathbf{c} \quad \text{and} \quad \mathbf{a} \times (\mathbf{b}' + \mathbf{c}') = \mathbf{a} \times (\mathbf{b} + \mathbf{c}).$$

Now $\mathbf{a}$ is perpendicular to $\mathbf{b}'$, so $\mathbf{a} \times \mathbf{b}'$ lies in π and is a times the length of $\mathbf{b}'$. Similarly, $\mathbf{a} \times \mathbf{c}'$ lies in π and is a times the length of $\mathbf{c}'$. Then it follows that $(\mathbf{a} \times \mathbf{b}' + \mathbf{a} \times \mathbf{c}')$ lies in π, is a times the length of $(\mathbf{b}'+\mathbf{c}')$, and is perpendicular to $(\mathbf{b}'+\mathbf{c}')$. Then

$$\mathbf{a} \times \mathbf{b}' + \mathbf{a} \times \mathbf{c}' = \mathbf{a} \times (\mathbf{b}' + \mathbf{c}'),$$

from which the result follows.

It is important to remember that

$$\mathbf{r} \times \mathbf{r} = \mathbf{0}$$

and that if $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\mathbf{k}}$ form the conventional right-handed rectangular triad, then

$$\hat{\jmath} \times \hat{\mathbf{k}} = \hat{\imath}, \quad \hat{\mathbf{k}} \times \hat{\imath} = \hat{\jmath}, \quad \hat{\imath} \times \hat{\jmath} = \hat{\mathbf{k}}, \quad \hat{\mathbf{k}} \times \hat{\jmath} = -\hat{\imath}, \quad \text{etc.}$$

It follows from this and from the distributive law that if the components of $\mathbf{b}$ and $\mathbf{c}$ are (b_x, b_y, b_z) and (c_x, c_y, c_z), then

$$\mathbf{b} \times \mathbf{c} = (b_y c_z - b_z c_y,\ b_z c_x - b_x c_z,\ b_x c_y - b_y c_x)$$

or, in determinant notation,

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{2.4.2}$$

The *triple scalar product* is defined by $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, and is often written as $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. If $\mathbf{a} = (a_x, a_y, a_z)$, then by (2.4.2),

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{2.4.3}$$

The absolute value of the triple scalar product is equal to the volume of the parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. A simple proof of this follows. In the notation of Figure 2.7 we have

$$\mathbf{b} \times \mathbf{c} = bc \sin\theta\, \hat{\mathbf{p}},$$

where $\hat{\mathbf{p}}$ is the unit vector perpendicular to $\mathbf{b}$ and $\mathbf{c}$, as shown. Then

$$\begin{aligned} [\mathbf{a}, \mathbf{b}, \mathbf{c}] &= bc \sin\theta\, \hat{\mathbf{p}} \cdot \mathbf{a} \\ &= bc \sin\theta\, a \cos\phi \\ &= \text{area of base times the perpendicular height} \\ &= \text{the volume of the parallepiped.} \end{aligned}$$

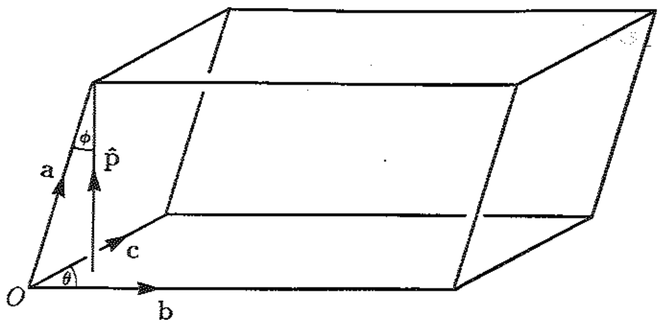

Figure 2.7

It follows from this or from the properties of determinants that

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}] = -[\mathbf{c},\mathbf{b},\mathbf{a}] = -[\mathbf{a},\mathbf{c},\mathbf{b}] = -[\mathbf{b},\mathbf{a},\mathbf{c}],$$

and that the triple scalar product of three nonzero vectors vanishes if any two are parallel or antiparallel or if all three vectors lie in a plane.

The *triple* or *continued vector product* is

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

in which the inclusion of parentheses is essential. It can be shown (most simply, if not most elegantly, by resolving the vectors) that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{2.4.4}$$

This formula is important and should be memorized.

Let P be any point on the line of action of a vector $\mathbf{Q}$, and let $\mathbf{OP} = \mathbf{r}$. Then the *moment* of $\mathbf{Q}$ about O is $\mathbf{r} \times \mathbf{Q}$. The reader should verify that this is independent of the position of P along $\mathbf{Q}$.

It is important to get the feel of the geometry of the vector product, as well as the algebra, for it is largely through this that vectors have such considerable practical use. Various uses of the product will appear shortly, but one will be pointed out now. If a vector equation contains some term that you do not like, you have only to multiply vectorially by its direction to eliminate it. For instance, the general equation of motion for a central orbit has the form

$$\frac{d^2\mathbf{r}}{dt^2} = -f\hat{\mathbf{r}}, \tag{2.4.5}$$

where f is some scalar function. Vectorial multiplication by $\mathbf{r}$ gives

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{0}, \tag{2.4.6}$$

which yields an important integral for any central orbit. Another example of the elimination of unwanted terms occurs in a consideration of the equation of the plane. This may be put in the form

$$\mathbf{r} = \mathbf{r}_0 + \lambda\hat{\mathbf{a}} + \mu\hat{\mathbf{b}}, \tag{2.4.7}$$

where $\mathbf{r}_0$ is any point in the plane and $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ lie in the plane. λ and μ are variable scalars that can be eliminated by scalar multiplication by $\hat{\mathbf{a}} \times \hat{\mathbf{b}}$. Then we have

$$(\mathbf{r} - \mathbf{r}_0)\cdot(\hat{\mathbf{a}} \times \hat{\mathbf{b}}) = 0. \tag{2.4.8}$$

Problems

1. Let $\hat{\imath}$ point toward the vernal equinox, $\hat{\mathbf{k}}$ toward the celestial north pole, $\hat{\mathbf{m}}$ toward the summer solstice, and $\hat{\mathbf{n}}$ toward the north pole of the ecliptic. Express $\hat{\imath}$ in terms of $\hat{\mathbf{n}}$ and $\hat{\mathbf{k}}$; $\hat{\mathbf{m}}$ in terms of $\hat{\mathbf{n}}$ and $\hat{\mathbf{k}}$; $\hat{\mathbf{n}}$ in terms of $\hat{\imath}$ and $\hat{\mathbf{k}}$; $\hat{\mathbf{k}}$ in terms of $\hat{\imath}$ and $\hat{\mathbf{n}}$.

2. Prove the formula for the triple vector product.

3. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, show that

$$\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{b},$$

and interpret geometrically.

4. What is the locus of $\mathbf{r}$ such that $[\mathbf{r},\mathbf{a},\mathbf{b}] = 0$?

5. If l, m, and n are three nonzero scalars, and

$$l\mathbf{a} + m\mathbf{b} + n\mathbf{c} = \mathbf{0},$$

show that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar. If also

$$l'\mathbf{a} + m'\mathbf{b} + n'\mathbf{c} = \mathbf{0},$$

show that

$$\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'}.$$

6. If $\mathbf{x}$, $\mathbf{a}$, and $\mathbf{b}$ are coplanar, show that usually $\mathbf{x}$ can be expressed as

$$\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b}.$$

What is the exception to this?

7. Evaluate $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$.

8. Choosing simple (but not trivial) numerical values for the components of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, verify numerically that

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}, \quad \mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c},$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

Find the values of

$$|\mathbf{a}\times\mathbf{c}|, \quad [\mathbf{a},\mathbf{b},\mathbf{c}], \quad (\mathbf{a}\times\mathbf{b})\times\mathbf{c}, \quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c}), \quad (\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c}).$$

9. Find the equation to the plane containing the heads of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

10. Show that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{a}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{a})$.

11. Show that

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= [\mathbf{a},\mathbf{b},\mathbf{d}]\mathbf{c} - [\mathbf{a},\mathbf{b},\mathbf{c}]\mathbf{d}\\ &= (\mathbf{c}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{a})\\ &= [\mathbf{c},\mathbf{d},\mathbf{a}]\mathbf{d} - [\mathbf{c},\mathbf{d},\mathbf{b}]\mathbf{a}.\end{aligned}$$

Hence show that

$$[\mathbf{a},\mathbf{b},\mathbf{c}]\mathbf{d} = [\mathbf{d},\mathbf{b},\mathbf{c}]\mathbf{a} + [\mathbf{a},\mathbf{d},\mathbf{c}]\mathbf{b} + [\mathbf{a},\mathbf{b},\mathbf{d}]\mathbf{c}$$

and deduce the conditions for the unique resolution of the vector $\mathbf{d}$ along the three directions $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

12. Show that

$$[\mathbf{a},\mathbf{b},\mathbf{c}](\mathbf{f}\times\mathbf{g}) = \begin{vmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\\ \mathbf{f}\cdot\mathbf{a} & \mathbf{f}\cdot\mathbf{b} & \mathbf{f}\cdot\mathbf{c}\\ \mathbf{g}\cdot\mathbf{a} & \mathbf{g}\cdot\mathbf{b} & \mathbf{g}\cdot\mathbf{c}\end{vmatrix}.$$

Hence, generalize equation (2.4.2).

13. Solve for $\mathbf{X}$ the equation $\mathbf{X}\times\mathbf{a} = \mathbf{b}$, where $\mathbf{a}\cdot\mathbf{b} = 0$.

14. Solve the equation $\mathbf{X} + \mathbf{a}(\mathbf{X}\cdot\mathbf{b}) = \mathbf{c}$. (Try $\cdot\,\mathbf{b}$.)

15. Solve the simultaneous equations

$$\alpha\mathbf{X} + \beta\mathbf{Y} = \mathbf{a} \quad\text{and}\quad \mathbf{X}\times\mathbf{Y} = \mathbf{b} \qquad \text{where } \mathbf{a}\cdot\mathbf{b} = 0.$$

16. Solve the equation $\mathbf{X}\times\mathbf{a} + (\mathbf{X}\cdot\mathbf{b})\mathbf{c} = \mathbf{d}$.

2.5 The Velocity of a Vector

A vector can vary as a function of a scalar, such as time, or as a function of another vector, such as position. Provided that the variation is suitably continuous (and this will normally be taken for granted), the vector can be differentiated. Here we shall mostly be considering vectors that vary with time, and the subject of differentiation will be approached in relation to the kinematics of a particle.

Let AB represent the path of a particle P, and let P and P' be positions of the particle at times t and $t+\delta t$. δt is a small time interval, and the arc PP' can be considered as linear because we are shortly to take the limit $\delta t \to 0$. Choose any origin O, and let $\mathbf{OP} = \mathbf{r}$ and $\mathbf{OP}' = \mathbf{r}+\delta\mathbf{r}$, so that $\mathbf{PP}' = \delta\mathbf{r}$. Let $\hat{\boldsymbol{\tau}}$ be the unit vector along $\mathbf{PP}'$; in the limit, this will be the tangent at P. Then

$$\delta\mathbf{r} = \hat{\boldsymbol{\tau}}\,\delta s$$

where s is distance measured along the curve. Taking the limit we have

$$\lim_{\delta t\to 0}\frac{\mathbf{PP}'}{\delta t} = \lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \hat{\boldsymbol{\tau}}\frac{ds}{dt}.$$

This is the *velocity* of the particle at P. It may also be written $\dot{\mathbf{r}}$ or $\mathbf{v}$. It has magnitude and direction, but we have to show that it obeys the vector law of addition, in order to establish that it is a vector.

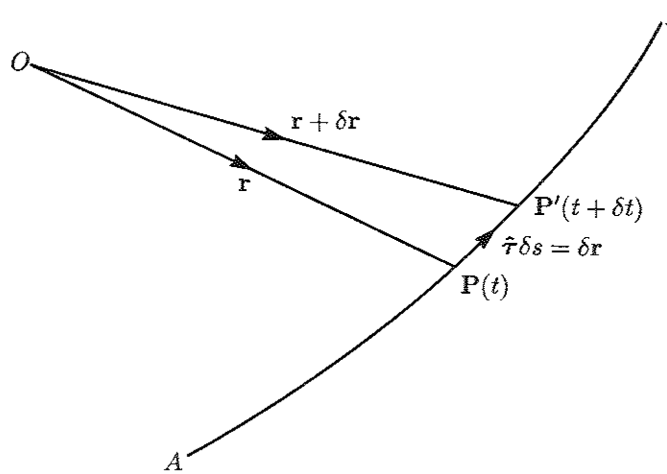

Figure 2.8

Let P have its position changed in time δt by two increments, $\delta_1\mathbf{r}$ and $\delta_2\mathbf{r}$. These are vectors, so that the total displacement is $\delta\mathbf{r} = \delta_1\mathbf{r}+\delta_2\mathbf{r}$. Dividing by

δt and taking the limit, we get

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}_1}{dt} + \frac{d\mathbf{r}_2}{dt},$$

which can be written as

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2.$$

This is the required result.

The components of $(\dot{\mathbf{r}})$ are $\dot{x}, \dot{y}, \dot{z}$ and those of $\hat{r}$ are $(dx/ds, dy/ds, dz/ds)$.

It is convenient to adopt the modern convention using "velocity" only as the vector $\mathbf{v}$, and "speed" as the scalar value of the velocity. Thus the speed of P is $\dfrac{ds}{dt}$.

It is very important to realize that $\left|\dfrac{d\mathbf{r}}{dt}\right|$, or $|\dot{\mathbf{r}}|$, is not in general the same as $\dfrac{d|\mathbf{r}|}{dt}$, or $\dfrac{dr}{dt}$, or $\dot{r}$. The former is the speed of P, while the latter is only the component of the velocity of P along the radius vector: in general there will also be a *transverse* component of velocity, at right angles to the radius vector. We have

$$\mathbf{r} \cdot \mathbf{r} = r^2.$$

Differentiating each side, we get

$$\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r}, \tag{2.5.1}$$

an important relation that will be used frequently in this text.

$\mathbf{r}$ is a position vector and the path traced out by P is the locus (or orbit, for our purposes) of P. $\dot{\mathbf{r}}$ is a localized vector, but if we let $\mathbf{OQ} = \dot{\mathbf{r}}$, then Q traces out a path called the *hodograph* of the motion. The rate of change of $\mathbf{OQ}$ measures the acceleration of P, which is a vector. It may be written $d^2\mathbf{r}/dt^2$, $d\mathbf{v}/dt$ or $\ddot{\mathbf{r}}$.

In Figure 2.8 consider the product $\mathbf{OP} \times \mathbf{OP'} = \mathbf{r} \times \delta\mathbf{r}$. The modulus is equal to twice the area of the triangle OPP', and the rate of change of this area is the *areal velocity* of OP. The areal velocity is therefore

$$\frac{1}{2}|\mathbf{r} \times \dot{\mathbf{r}}|.$$

The direction of $\mathbf{r} \times \dot{\mathbf{r}}$ is perpendicular to the plane containing $\mathbf{r}$ and $\dot{\mathbf{r}}$. If the motion takes place in a plane, then this direction is constant. Consider Kepler's first two laws: the first specifies motion in a plane for a planet, and the second states that the areal velocity is constant. So, for Keplerian motion, we have

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}, \tag{2.5.2}$$

where $\mathbf{h}$ is a constant vector. Note that (2.5.2) is the first integral of (2.4.6) and that this means that the acceleration is entirely directed along $\hat{\mathbf{r}}$.

Problems

1. Solve the following equations of motion, and interpret their solutions.

$$\text{(a) } \dot{\mathbf{r}} = \mathbf{a} \quad \text{(b) } \ddot{\mathbf{r}} = \mathbf{b} \quad \text{(c) } \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{0} \quad \text{(d) } \mathbf{r} \cdot \dot{\mathbf{r}} = c$$
$$\text{(e) } |\dot{\mathbf{r}}| = \dot{r} \quad \text{(f) } \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} \quad \text{(g) } \dot{\mathbf{r}} = \mathbf{r} \times \ddot{\mathbf{r}}.$$

2. Show that $\dfrac{d(a\mathbf{r})}{dt} = \mathbf{r}\dfrac{da}{dt} + a\dfrac{d\mathbf{r}}{dt}$.

3. Show that $\dfrac{d(\mathbf{a} \cdot \mathbf{b})}{dt} = \mathbf{a} \cdot \dfrac{d\mathbf{b}}{dt} + \mathbf{b} \cdot \dfrac{d\mathbf{a}}{dt}$.

4. Show that $\dfrac{d(\mathbf{a} \times \mathbf{b})}{dt} = \mathbf{a} \times \dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt} \times \mathbf{b}$.

5. Describe the hodograph of uniform circular motion with radius a and speed v. Hence show that the acceleration is (v^2/a) directed toward the center.

6. A particle describes an equiangular spiral with constant areal velocity A. Show that for the motion $r\dot{r} = 2A\cot\theta$, where θ is the (constant) angle between the radius vector and the tangent, and solve this equation to show how r varies with the time.

2.6 Angular Velocity

A vector can change its direction as well as its modulus, so that it is possible to differentiate a unit vector, $\hat{\imath}$, say. Let $\hat{\imath}$ be rotated through a small angle, $\delta\theta$, the new vector being $\hat{\imath} + \delta\hat{\imath}$. Since this must still be a unit vector, $\hat{\imath} \cdot \delta\hat{\imath} = 0$.

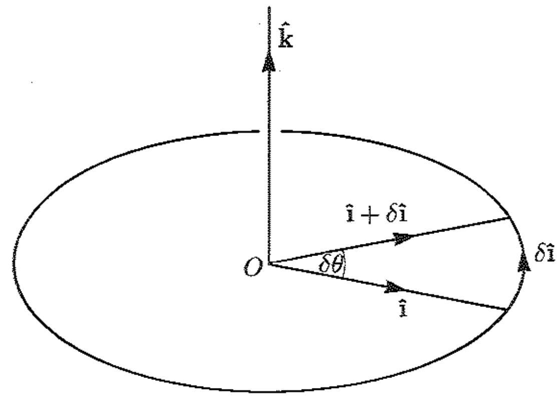

Figure 2.9

(Squares of small quantities, such as $\delta\hat{\imath}^2$, are, of course, neglected.) Let the direction of the rotation, taken in the right-handed sense, be $\hat{\mathbf{k}}$. Then the direction of $\delta\hat{\imath}$ is $\hat{\mathbf{k}} \times \hat{\imath}$ and its length is $\delta\theta$, so

$$\delta\hat{\imath} = \delta\theta\hat{\mathbf{k}} \times \hat{\imath},$$

and

$$\begin{aligned}\frac{d\hat{\imath}}{dt} &= \left(\hat{\mathbf{k}}\frac{d\theta}{dt}\right) \times \hat{\imath} \\ &= \boldsymbol{\omega} \times \hat{\imath},\end{aligned}$$

where $\boldsymbol{\omega}$ is an *angular velocity*. It is useful to remember that the derivative of a unit vector is always perpendicular to that vector.

More generally, let $\mathbf{OP} = \mathbf{r}$ be any vector, making an angle θ with $\hat{\mathbf{k}}$, the axis of rotation. Then, since P is at a distance $r \sin\theta$ from $\hat{\mathbf{k}}$, the rate of change of $\mathbf{r}$ due *solely* to the angular velocity $\boldsymbol{\omega}$ is $\boldsymbol{\omega} \times \mathbf{r}$.

$\boldsymbol{\omega}$ has magnitude and direction. To show that it is a vector, we must prove that it obeys the vector addition law. Suppose $\mathbf{r}$ to be subjected to two infinitesimal rotations, $\delta\theta_1$ about $\hat{\mathbf{k}}_1$ and $\delta\theta_2$ about $\hat{\mathbf{k}}_2$. Taking these in order, we have $\mathbf{r}$ becoming first

$$\mathbf{r} + \delta\mathbf{r}_1 = \mathbf{r} + \delta\theta_1\hat{\mathbf{k}}_1 \times \mathbf{r}$$

and then

$$\mathbf{r} + \delta\mathbf{r}_{12} = (\mathbf{r} + \delta\theta_1\hat{\mathbf{k}}_1 \times \mathbf{r}) + \delta\theta_2\hat{\mathbf{k}}_2 \times (\mathbf{r} + \delta\theta_1\hat{\mathbf{k}}_1 \times \mathbf{r}).$$

Neglecting the second order small quantity $\delta\theta_1\delta\theta_2$ (this is all right since we are to take a limit; it is this step which is not possible when dealing with finite rotations), we have

$$\delta\mathbf{r}_{12} = \delta\theta_1\hat{\mathbf{k}}_1 \times \mathbf{r} + \delta\theta_2\hat{\mathbf{k}}_2 \times \mathbf{r} = \delta\mathbf{r}_{21},$$

showing that infinitesimal rotations are commutative. Dividing by δt and taking the limit, we see that the same is true for angular velocities. If

$$\hat{\mathbf{k}}_1\frac{d\theta_1}{dt} = \boldsymbol{\omega}_1 \quad \text{and} \quad \hat{\mathbf{k}}_2\frac{d\theta_2}{dt} = \boldsymbol{\omega}_2$$

then

$$\dot{\mathbf{r}} = (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \mathbf{r},$$

so that the effect is the same as that produced by a single angular velocity $(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)$. Hence angular velocities are vectors; they can be added according to the vector law and they can also be resolved.

Problems

1. Find expressions for the modulus and direction of the angular velocity of the mean Sun.

2. Find the angular velocity of the apparent Sun, assuming that the Earth's orbit is circular, with constant angular velocity. Resolve this angular velocity along an equatorial system of axes and hence find the components of the Sun's motion on the celestial sphere along these axes. Compare the component in right ascension with the angular velocity of the mean Sun.

3. Assuming the Earth to be fixed and the celestial sphere (of unit radius) to be rotating around the Earth, consider the apparent motion of the stars as seen by an observer at latitude l. If $\hat{\mathbf{z}}$ points to the observer's zenith (the direction vertically above him), resolve the motion of a star at declination δ along and at right angles to $\hat{\mathbf{z}}$.

2.7 Rotating Axes

Writing $\mathbf{r} = r\hat{\mathbf{r}}$ and differentiating, we have

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}. \tag{2.7.1}$$

This expresses $\dot{\mathbf{r}}$ as the sum of two vectors, one along and one perpendicular to $\mathbf{r}$. The first gives the rate of change of the length of $\mathbf{r}$, and the second can be considered as being due to a rotation.

Frequently (as on the Earth) we observe phenomena with respect to rotating axes. Suppose a vector $\mathbf{r}$ to be observed to have rates of change $\frac{d\mathbf{r}}{dt}$ with respect to a fixed frame of reference F_1, and $\frac{\partial\mathbf{r}}{\partial t}$ with respect to a frame F_2, rotating with respect to F_1 with angular velocity $\boldsymbol{\omega}$, which need not be constant. The formula (2.7.1) applies to a simple case of this, where the axes of F_2 are rotating with $\mathbf{r}$, so that $\frac{\partial\mathbf{r}}{\partial t}$ can be written instead of $\dot{r}\hat{\mathbf{r}}$. Then (2.7.1) can be written in the form

$$\frac{d\mathbf{r}}{dt} = \frac{\partial\mathbf{r}}{\partial t} + \boldsymbol{\omega} \times \mathbf{r}. \tag{2.7.2}$$

This is a general result, as we shall now show.

Let $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\mathbf{k}}$ form an orthogonal triad rigidly attached to F_2; then

$$\mathbf{r} = \sum(\mathbf{r}\cdot\hat{\imath})\hat{\imath},$$

where the summation is over $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\mathbf{k}}$. Then

$$\frac{d\mathbf{r}}{dt} = \sum\left\{\frac{d(\mathbf{r}\cdot\hat{\imath})}{dt}\hat{\imath}\right\} + \sum\left\{(\mathbf{r}\cdot\hat{\imath})\frac{d\hat{\imath}}{dt}\right\}.$$

But

$$\frac{d\hat{\imath}}{dt} = \boldsymbol{\omega} \times \hat{\imath}$$

so

$$\sum\left\{(\mathbf{r}\cdot\hat{\imath})\frac{d\hat{\imath}}{dt}\right\} = \boldsymbol{\omega}\times\sum(\mathbf{r}\cdot\hat{\imath})\hat{\imath}$$
$$= \boldsymbol{\omega}\times\mathbf{r}.$$

To find $\dfrac{\partial\mathbf{r}}{\partial t}$ we treat $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\mathbf{k}}$ as constant vectors, so that

$$\frac{\partial\mathbf{r}}{\partial t} = \sum\left\{\frac{d(\mathbf{r}\cdot\hat{\imath})}{dt}\hat{\imath}\right\}.$$

Hence (2.7.2) follows at once. It may be written in the notation of operators as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{\omega}\times . \tag{2.7.3}$$

As an illustration of the use of this formula we shall find the components of velocity and acceleration in polar coordinates. Here $\mathbf{OP} = \mathbf{r}$ makes an angle θ with some fixed direction. Let $\hat{\imath}$ point in the *radial* direction (along $\mathbf{OP}$) and $\hat{\jmath}$ in the *transverse* direction (i.e., in the direction of increasing θ), and let $\hat{\imath}\times\hat{\jmath} = \hat{\mathbf{k}}$. (See Figure 2.10.) Then the angular velocity of $\mathbf{OP}$ is $\dot{\theta}\hat{\mathbf{k}}$, and

$$\frac{d\mathbf{r}}{dt} = \dot{r}\hat{\imath} + r\dot{\theta}\hat{\jmath}. \tag{2.7.4}$$

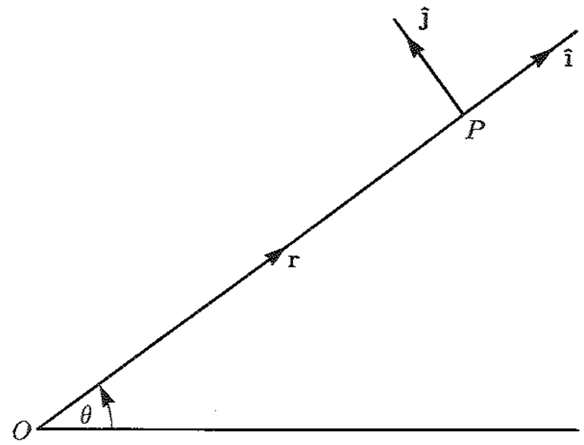

Figure 2.10

This gives the radial and transverse components of velocity. Similarly,

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d}{dt}\right)(\dot{r}\hat{\imath} + r\dot{\theta}\hat{\jmath})$$

$$= \left(\frac{\partial}{\partial t} + \dot{\theta}\hat{\mathbf{k}}\times\right)(\dot{r}\hat{\imath} + r\dot{\theta}\hat{\jmath})$$
$$= \ddot{r}\hat{\imath} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\jmath} + \dot{\theta}\hat{\mathbf{k}}\times(\dot{r}\hat{\imath} + r\dot{\theta}\hat{\jmath})$$
$$= (\ddot{r} - r\dot{\theta}^2)\hat{\imath} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\jmath}. \tag{2.7.5}$$

This formula, which it is useful to memorize, gives the radial and transverse components of acceleration. The transverse component can be written as

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

The term in the parentheses is easily verified to be twice the areal velocity of P.

Consider the situation when a particle travels in a circle with constant speed. Taking the center of the circle as origin, we have

$$\frac{d^2\mathbf{r}}{dt^2} = -r\dot{\theta}^2\hat{\imath}.$$

This acceleration is a consequence of the kinematics of the particle. If an observer is moving round with the particle, and, through ignorance or prejudice or for convenience, he likes to work in terms of the kinematics with respect to his rotating axes, then he will be unable to account for what he observes unless he applies to every particle a "centrifugal acceleration" $r\dot{\theta}^2\hat{\imath}$. Alternatively he can introduce a fictitious force, the "centrifugal force," which would produce this acceleration.

As an example of its application consider a conical pendulum consisting of a mass m at P suspended from O by a light string of length r, and traveling in a circle with constant angular velocity about the vertical such that OP makes a constant angle ϕ with the vertical. The radius of the circle is $r\sin\phi$, so that the centrifugal force C, acting as shown in Figure 2.11 is equal to $mr\sin\phi\,\dot{\theta}^2$, where $\dot{\theta}$ is the angular velocity of P about the vertical. With this force included, the problem becomes one of statics. Resolving the forces at right angles to the string (to eliminate the tension), we have

$$mg\sin\phi = C\cos\phi = mr\sin\phi\cos\phi\,\dot{\theta}^2$$

so

$$\dot{\theta}^2 = \frac{g}{r\cos\phi}.$$

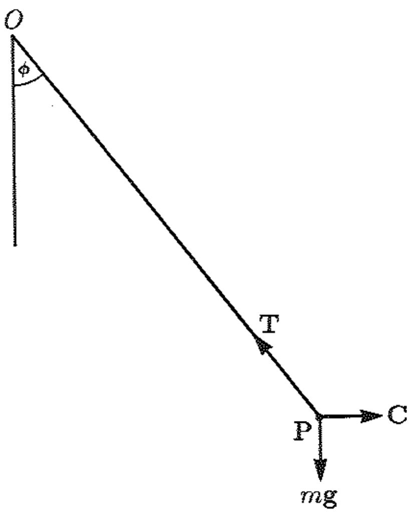

Figure 2.11

As another illustration of the use of equation (2.7.3) consider the motion observed from a point fixed on the surface of the Earth. Let C be the center of the Earth, and let someone at O be observing the motion of a point P. Let $\mathbf{CO} = \mathbf{r}_0$ and $\mathbf{OP} = \mathbf{r}$; usually r is small compared with r_0. Let the angular velocity of the Earth be $\omega\hat{\mathbf{z}}$. The velocity of P with respect to nonrotating axes is

$$\begin{aligned}\frac{d\mathbf{P}}{dt} &= \frac{d}{dt}(\mathbf{r}_0 + \mathbf{r})\\ &= \frac{\partial \mathbf{r}}{\partial t} + \omega\hat{\mathbf{z}} \times \mathbf{r}_0 + \omega\hat{\mathbf{z}} \times \mathbf{r}.\end{aligned}$$

The acceleration of P is

$$\begin{aligned}\frac{d^2\mathbf{P}}{dt^2} &= \left(\frac{\partial}{\partial t} + \omega\hat{\mathbf{z}}\times\right)\frac{d\mathbf{P}}{dt}\\ &= \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\omega\hat{\mathbf{z}} \times \frac{\partial \mathbf{r}}{\partial t} + \omega^2\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \mathbf{r}_0) + \omega^2\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \mathbf{r}).\end{aligned}$$

The usual equation of motion of P will contain $d^2\mathbf{P}/dt^2$ and terms dealing with the forces acting on P. If the equation is considered in terms of $\partial^2\mathbf{P}/\partial t^2$, or $\partial^2\mathbf{r}/\partial t^2$, or in terms of what the person at O observes, then we can replace $d^2\mathbf{P}/dt^2$ on one side of the equation by $\partial^2\mathbf{P}/\partial t^2$, provided we add the terms

$$-2\omega\hat{\mathbf{z}} \times \frac{\partial \mathbf{r}}{\partial t} - \omega^2\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \mathbf{r}_0) - \omega^2\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \mathbf{r}) \qquad (2.7.6)$$

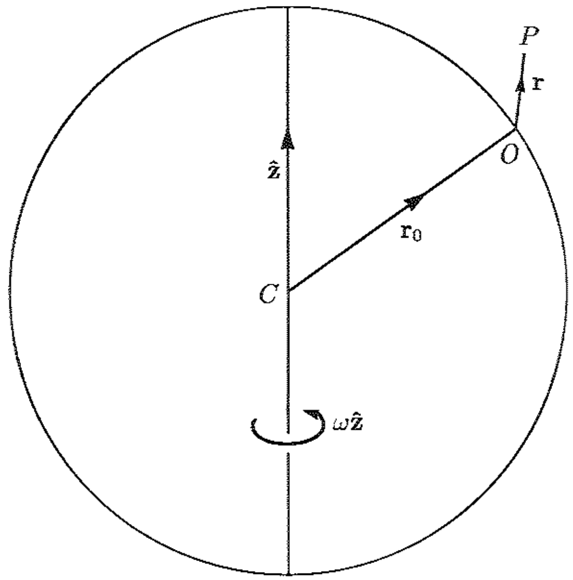

Figure 2.12

to the other side, among the terms dealing with the forces acting on P. (Usually the last two terms can be neglected, since ω is small.) Again, the terms in (2.7.6) can be interpreted as being due to fictitious forces: the first is known as the *Coriolis force*; the final two make up the centrifugal force. An example concerning them will be given in Section 3.8.

Problems

1. What conditions are necessary for $\ddot{r} = |\ddot{\mathbf{r}}|$?

2. If P moves in a plane and its areal velocity about O is constant, show that the transverse acceleration of P is zero.

3. Solve the simultaneous equations

$$\ddot{r} - r\dot{\theta}^2 = 0 \quad \text{and} \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0.$$

   What can be deduced from these?

4. A particle moves in an ellipse with semiaxes a and b, with constant areal velocity A about the center of the ellipse. Find the components of its velocity and acceleration in cartesian and polar coordinates. Find also the period of the orbit.

2.8 The Gradient of a Scalar

Consider a scalar function of position. This may be written $f(x, y, z)$ or $f(\mathbf{r})$. We shall suppose that it is defined and is differentiable within the region of space with which we are concerned. If we move from (x, y, z), to $(x + \delta x, y, z)$, the change in f can be written $(\partial f/\partial x)\delta x$, where now $\partial/\partial x$ represents conventional partial differentiation with respect to x, so that for this differentiation, y and z are assumed to be constant. This could be written in the form

$$\begin{aligned}\frac{\partial f}{\partial x}\delta x &= \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) \cdot \hat{\imath}\,\delta x \\ &= \nabla f \cdot \hat{\imath}\,\delta x.\end{aligned}$$

Similarly, a change from (x, y, z) to $(x, y + \delta y, z)$ produces a change in f of $\nabla f \cdot \hat{\jmath}\,\delta y$, and a change from $\mathbf{r}$ to $\mathbf{r} + \delta\mathbf{r}$ produces a change $\nabla f \cdot \delta\mathbf{r}$.

The vector with components $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ is called the *gradient* of f. It is written as "grad f" or "∇f"; the symbol "∇" is called "Nabla" or "Del." Since

$$\frac{\partial}{\partial x}(f_1 + f_2) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x}$$

there is no difficulty in establishing that ∇f obeys the vector law of addition; hence it is definitely a vector.

Consider a curve C and the values that f takes along this curve. The rate of change of f along C with respect to arc length s is

$$\begin{aligned}\frac{df}{ds} &= \frac{\partial f}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial f}{\partial y}\cdot\frac{dy}{ds} + \frac{\partial f}{\partial z}\cdot\frac{dz}{ds} \\ &= \hat{\boldsymbol{\tau}} \cdot \nabla f, \qquad (2.8.1)\end{aligned}$$

from Section 2.5. Consider a surface with equation $f(x, y, z) =$ constant. Along any line on the surface $df/ds = 0$, so that $\hat{\boldsymbol{\tau}} \cdot \nabla f = 0$. Therefore ∇f is perpendicular to the surface.

∇f is an example of a *field vector*, i.e., a vector that is a function of position.

Problems

1. Prove that $\nabla r^n = n\mathbf{r}r^{n-2}$.
2. Show that the components of ∇ in two-dimensional polar coordinates are
$$\frac{\partial}{\partial r} \quad \text{and} \quad \frac{1}{r}\cdot\frac{\partial}{\partial \theta}.$$
3. Find the components of ∇ in cylindrical coordinates (r, ϕ, z) and in spherical polar coordinates (r, θ, ϕ).
4. If f and g are scalar functions of $\mathbf{r}$, show that
$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}.$$

2.9 Spherical Trigonometry

We conclude this chapter with a short account of basic spherical trigonometry approached through vectors.

Spherical trigonometry is set on the surface of a sphere that is assumed to have unit radius. All lines with which we are concerned are portions of great circles. If two points are not at opposite ends of a diameter, then there is one and only one great circle passing through them. A spherical triangle ABC is defined by three points on the surface of the sphere, the sides of the triangle being the appropriate parts of the great circles through the points. If O is the center of the sphere,

$$\angle BOC = a, \quad \angle COA = b, \quad \text{and} \quad \angle AOB = c.$$

The angle between the planes AOB and AOC is A, and B and C are similarly defined.

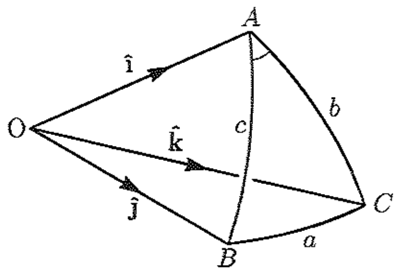

Figure 2.13

Let $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\mathbf{k}}$ be the unit vectors (not mutually perpendicular) along **OA**, **OB**, and **OC**. $(\hat{\imath} \times \hat{\jmath})$ is a vector with magnitude $\sin c$ and direction perpendicular to the plane AOB. Similarly, $(\hat{\imath} \times \hat{\mathbf{k}})$ has magnitude $\sin b$ and direction perpendicular to the plane AOC. So the angle between these two vectors is A. Hence

$$(\hat{\imath} \times \hat{\jmath}) \cdot (\hat{\imath} \times \hat{\mathbf{k}}) = \sin b \sin c \cos A.$$

But

$$\begin{aligned}(\hat{\imath} \times \hat{\jmath}) \cdot (\hat{\imath} \times \hat{\mathbf{k}}) &= \hat{\imath} \cdot [\hat{\jmath} \times (\hat{\imath} \times \hat{\mathbf{k}})] \\ &= \hat{\imath} \cdot [\hat{\imath}(\hat{\jmath} \cdot \hat{\mathbf{k}}) - \hat{\mathbf{k}}(\hat{\imath} \cdot \hat{\jmath})] \\ &= (\hat{\jmath} \cdot \hat{\mathbf{k}}) - (\hat{\imath} \cdot \hat{\mathbf{k}})(\hat{\imath} \cdot \hat{\jmath}) \\ &= \cos a - \cos c \cos b.\end{aligned}$$

Hence

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad (2.9.1)$$

By the definition of the vector product,

$$\begin{aligned}\sin A &= \frac{|(\hat{\imath}\times\hat{\jmath})\times(\hat{\imath}\times\hat{\mathbf{k}})|}{|\hat{\imath}\times\hat{\jmath}||\hat{\imath}\times\hat{\mathbf{k}}|}\\ &= \frac{|-\hat{\imath}[\hat{\jmath},\hat{\imath},\hat{\mathbf{k}}]+\hat{\jmath}[\hat{\imath},\hat{\imath},\hat{\mathbf{k}}]|}{\sin b\sin c}.\\ &= \frac{[\hat{\imath},\hat{\jmath},\hat{\mathbf{k}}]}{\sin b\sin c}.\end{aligned}$$

Hence

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}=\frac{6\ \mathrm{vol}(OABC)}{\sin a\sin b\sin c}. \tag{2.9.2}$$

These equations are the fundamental formulas of spherical trigonometry. Two other formulas are frequently used; they can be derived from (2.9.1) and (2.9.2) by elementary trigonometry. They are

$$\sin a\cos B=\cos b\sin c-\sin b\cos c\cos A \tag{2.9.3}$$

and

$$\cos a\cos C=\sin a\cot b-\sin C\cot B. \tag{2.9.4}$$

As an illustration of the use of these formulas, consider the problem of expressing right ascension and declination in terms of celestial longitude and latitude.

Let X be a star at (α,δ) or (λ,β). Let N and P be the celestial north pole and the north pole of the ecliptic, and let the great circles through these points and X meet the celestial equator and ecliptic in A and B, respectively, as shown in Figure 2.14, where the great circles through $P\Upsilon$ and $N\Upsilon$ are also drawn. Υ is the pole of the great circle PN, so the angles $PN\Upsilon$ and $NP\Upsilon$ are both equal to 90°.

Now

$$\Upsilon A=\angle\Upsilon NA=\alpha$$

so

$$\angle PNX=90^\circ+\alpha.$$

Similarly,

$$\angle NPX=90^\circ-\lambda.$$

Also,

$$PN=\epsilon,$$
$$NX=90^\circ-\delta,$$

and

$$PX=90^\circ-\beta.$$

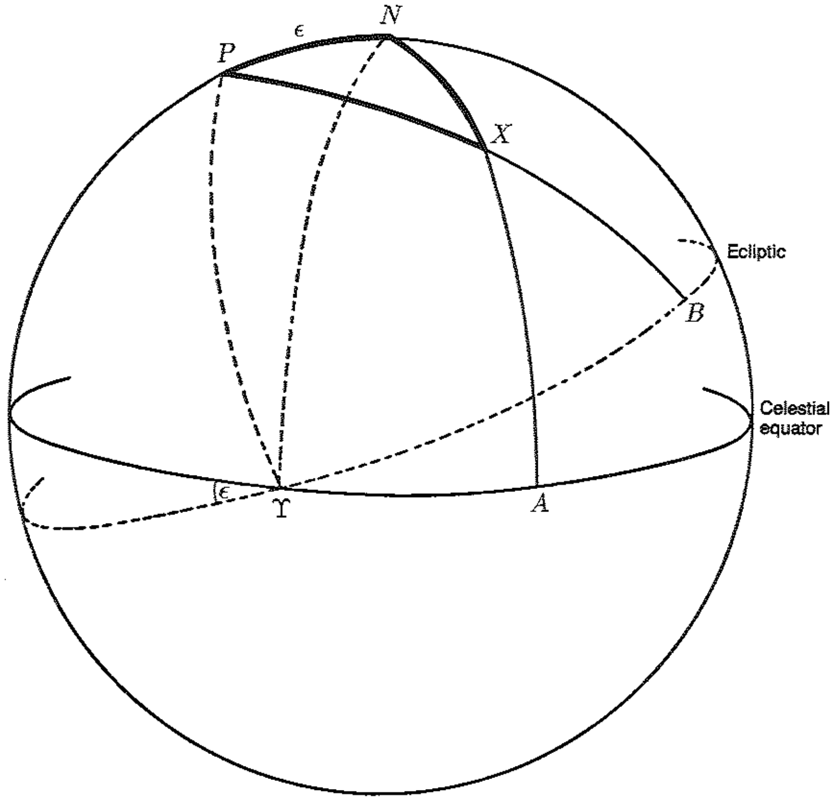

Figure 2.14

Applying the formulas (2.9.1), (2.9.2), and (2.9.3) to the triangle PNX, we have

$$\begin{aligned}\cos PX &= \cos NX\cos PN+\sin NX\sin PN\cos PNX,\\ \sin PX\sin NPX &= \sin NX\sin PNX,\\ \sin PX\cos NPX &= \cos NX\sin PN-\sin NX\cos PN\cos PNX,\end{aligned}$$

or

$$\begin{aligned}\sin\beta &= \sin\delta\cos\epsilon-\cos\delta\sin\epsilon\sin\alpha,\\ \cos\beta\cos\lambda &= \cos\delta\cos\alpha,\\ \cos\beta\sin\lambda &= \sin\delta\sin\epsilon+\cos\delta\cos\epsilon\sin\alpha.\end{aligned} \tag{2.9.5}$$

These expressions can also be found quickly without using spherical trigonometry. If we take axes, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\mathbf{k}}$, with $\hat{\imath}$ pointing toward Υ and $\hat{\mathbf{k}}$ toward

N, and let these be changed to $\hat{\mathbf{l}}$, $\hat{\mathbf{m}}$, and $\hat{\mathbf{n}}$ by a rotation about $O\Upsilon$ through the angle ϵ, so that $\hat{\mathbf{n}}$ points toward P, $\hat{\mathbf{m}}$ lies in the ecliptic (it is the summer solstice) and $\hat{\mathbf{i}} = \hat{\mathbf{l}}$, then

$$\hat{\mathbf{m}} = \hat{\mathbf{j}} \cos\epsilon + \hat{\mathbf{k}} \sin\epsilon,$$
$$\hat{\mathbf{n}} = -\hat{\mathbf{j}} \sin\epsilon + \hat{\mathbf{k}} \cos\epsilon.$$

But the components of OX along these two sets of axes are

$$(\cos\delta\cos\alpha,\ \cos\delta\sin\alpha,\ \sin\delta)$$

and

$$(\cos\beta\cos\lambda,\ \cos\beta\sin\lambda,\ \sin\beta),$$

and equations (2.9.5) follow at once. So, although it is useful to be familiar with the methods of spherical trigonometry, you should be on the lookout for methods that are quicker and perhaps safer.

Rotation matrices can often be used in preference to spherical trigonometry, especially in computations. The following is an elegant derivation of several formulas of spherical trigonometry using rotation matrices. (I am indebted to B.G. Marsden for this method.) First, make certain that you thoroughly understand the matrices. (See Appendix B.) For the triangle of Figure 2.13, start with axes such that the xy-plane is in the plane OBC and the x-axis points along **OB**. The sequence of rotations: $R(a)$, $P(-C)$, $R(-b)$, will result in the new x-axis pointing along **OA** with the new y-axis on the great circle AC. The identical result can be generated using the rotations: $P(B), R(c), P(A-\pi)$. So we have the identity

$$R(-b)P(-C)R(a) = P(A-\pi)R(c)P(B).$$

Substitute, multiply the matrices, and then compare components!

Problem

Using the triangle PNX, derive expressions for α and δ in terms of λ and β. Check these expressions, using the alternative method indicated above.

Chapter 3

Introduction To Vectorial Mechanics

3.1 Forces as Vectors

Mechanics deals with the effects of forces, and the first step in this chapter is to establish that forces are vectors. Certainly forces have magnitudes and directions, so we have to show that they can be added or resolved according to the vector law. It can be shown experimentally that a force P can be resolved in a direction making θ with its line of action, and that the resultant is $P\cos\theta$; so, forces can be resolved and therefore added as vectors. Alternatively we can accept the properties of the triangle of forces to be experimentally proved. That is, if three forces acting through a point are in equilibrium, then it is possible to construct a triangle with sides parallel to the lines of action of the forces and with lengths proportional to their magnitudes. This is precisely equivalent to the vector law of addition.

Forces are localized vectors, and they must always be treated as such. Expressions such as "resolving a force" or "the moment of a force" follow from their vector definitions.

3.2 Basic Definitions

The *mass* of a body is a measure of the amount of material in the body. A unit of mass can be defined in terms of a definite volume of some standard substance; for instance, the gram is the mass of a cubic centimeter of water. At some point on the Earth's surface the force exerted on a body of mass m is mg, where g is constant for that place, due mostly to the Earth's gravity. mg is the *weight* of the body. On the Moon the body would weigh less, but its mass would remain the same; hence mass, and not weight, is the fundamental quantity. Two masses can be compared at some place by comparing their weights, so there is no difficulty in allotting a measure to any particular mass. Mass is, of course, a scalar quantity. In the work immediately following, it will be convenient to assume that bodies are *point masses*; a point mass has its entire mass concentrated at a geometrical point.