5. A body free to turn about its center of gravity, which is fixed, is in stable equilibrium under the attraction of a distant fixed particle M' at distance R. Show that the axis of least moment is turned toward the particle. Show also that the times of the principle oscillations are

$$2\pi\left\{\frac{BR^3}{3M'G(C-A)}\right\}^{1/2} \quad \text{and} \quad 2\pi\left\{\frac{CR^3}{3M'G(B-A)}\right\}^{1/2}$$

If the body be the Earth, and M' the Sun, show that the smaller of these two periods is about ten years.

Appendix A

Properties of Conics

A.1 General Properties

The word "conic"comes from the phrase "conic section." Suppose that we have a circle and a point V which is not in the plane of the circle; then lines containing V and points on the circle describe a cone. Any plane section of this cone will be a conic, and any conic can be constructed in this way. (See Figure A.1.)

The graph of the general equation of second degree in cartesian coordinates is (if there are real points satisfying the equation) a conic. Conics can be defined in this way, and their properties discussed through the properties of quadratic forms. One property is that by a transfer of origin and a rotation of the axes, we can *usually* reduce the equation to the form

$$ax^2 + by^2 + c = 0. \tag{A.1.1}$$

If this is possible (one exception is the parabola) then the origin of coordinates is called the *center* of the conic, and the coordinate axes are the *axes* of the conic. For work in cartesian coordinates it is usually simplest to use these axes.

Another important definition can be derived as follows. Pick one of the conic sections from Figure A.1 and insert between the section and the vertex the largest possible sphere; this will touch the cone in a circle, C. Let the plane containing the circle be called Π. The line of intersection of Π and the plane of the conic section is called a *directrix* of the conic. Let the sphere touch the plane of the conic section at a point S; this is a focus of the conic. See Figure A.2.

Let P be a point on the conic, and let the line VP intersect the circle C at the point A. The lines PS and PA touch the sphere at S and A, and therefore $PS = PA$. Now drop a perpendicular from P onto the plane Π, meeting the plane at Q. Let the *half*-angle of the cone be θ. Then from the triangle PQA we have $AP = PQ\sec\theta$. Let the angle between Π and the plane of section be ϕ, and let R be a point on the directrix such that PR is perpendicular to

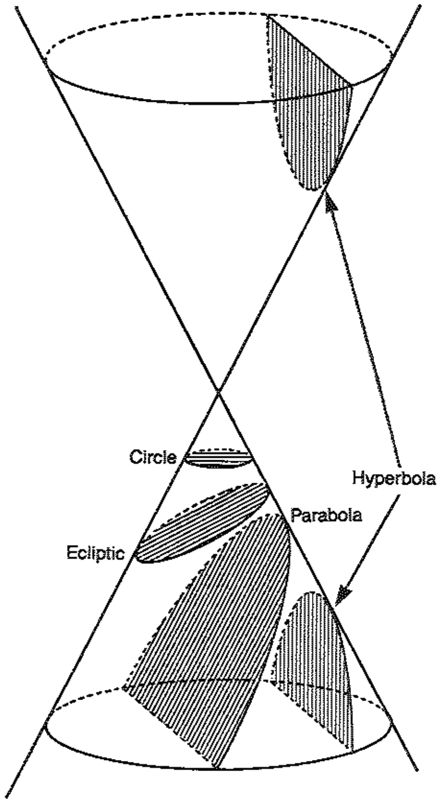

Figure A.1

the directrix. Then from the triangle PQR, $PQ = PR\sin\phi$. Combining these results, $PS = (\sec\theta\sin\phi)PR$. These constructions are illustrated in Figure A.2. The result can be put into words as follows: a conic is the set of points P such that the ratio of the distance of P from a fixed point (a focus) to the distance of P from a fixed line (a directrix) is constant.

To find the equation of the conic in polar coordinates, choose the focus as origin, and let the polar angle, v, be measured from the line OD perpendicular to the directrix. See Figure A.3. Then if $P(r, v)$ is a point on the conic,

$$r = e(k - r\cos v)$$

where k is the distance of the origin from the directrix and e is the constant of proportionality: it turns out to be the eccentricity of the conic. If $p = ek$, called the *parameter* of the conic, then the equation can be put into the form

$$\frac{p}{r} = 1 + e\cos v. \tag{A.1.2}$$

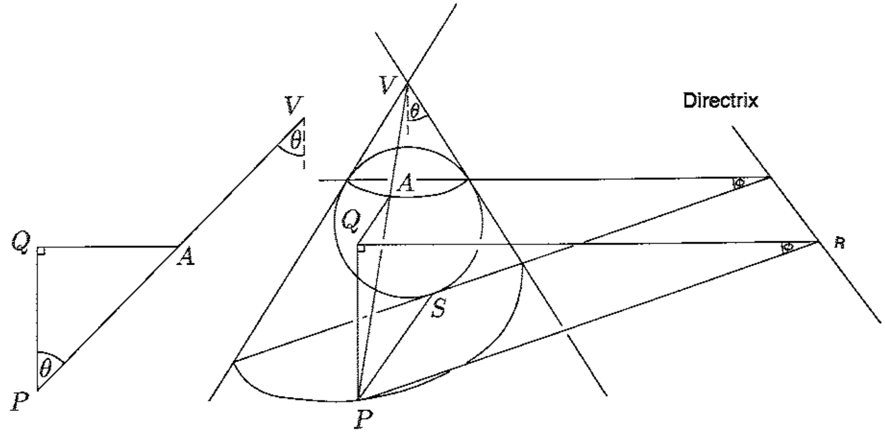

Figure A.2

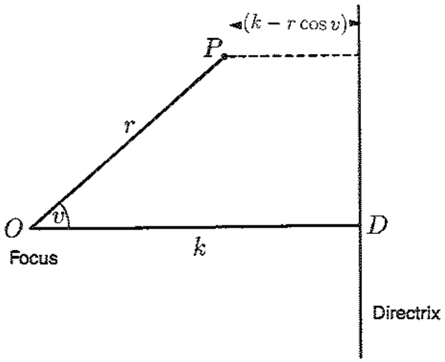

Figure A.3

The line OD, being an axis of symmetry, is an axis of the conic. The minimum value of r occurs when $v = 0$, and the maximum, if it exists, when $v = \pi$. The chord through O, perpendicular to OD, is called the *latus rectum* and has length $2p$. p is often referred to as the *semilatus rectum*. (In the constructions of Apollonius (c.200 B.C.) a rectangle was formed based on a line tangent to the conic and sticking out from the cone; this line has length $2p$. It was called ὀρθία, meaning *erect*, and the original Latin translation was *latus erectum*. The phrase used today is a corruption of this.)

A.2 The Ellipse

The ellipse is a conic for which $0 \le e < 1$. If $e = 0$, the ellipse becomes a circle. From the polar equation we see that r is bounded, so that the ellipse is a closed figure. It follows from the theory of quadratic forms that the cartesian equation of an ellipse can be put into the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{A.2.1}$$

a and b are positive, and a is invariably chosen to be greater than or equal to b. The x- and y-axes are the major and minor axes of the ellipse, respectively, and a and b are the *semimajor* and *semiminor axes.*

From an examination of the polar equation, (A.1.2), we see that the focus used as origin lies on the x-axis, and that the directrix is parallel to the y-axis. Let the focus be S; then, by symmetry, there must be another focus S' such that the center C bisects SS'. Let the axes meet the ellipse at A, A', B, and B' (see Figure A.4); then it is easily verified that

$$\begin{aligned}
CA &= CA' = a,\\
CB &= CB' = b,\\
SA &= q = a(1-e),\\
SA' &= q' = a(1+e),\\
CS &= CS' = ae,\\
p &= a(1-e^2),\\
b^2 &= a^2(1-e^2),\\
SB &= a.
\end{aligned}$$

The polar equation can be written

$$\frac{a(1-e^2)}{r} = 1 + e\cos v. \tag{A.2.2}$$

These formulas should all be memorized.

Another definition of the ellipse is that it is the locus of P such that

$$SP + S'P = \text{constant},$$

where S and S' are fixed points; clearly these are the foci, and the constant has the value $2a$. To see this, refer to Figure A.5 where two spheres have been inserted into the cone, one above and one below the ellipse; these touch the ellipse at points S and S', and the cone in circles C and C'. Let P be

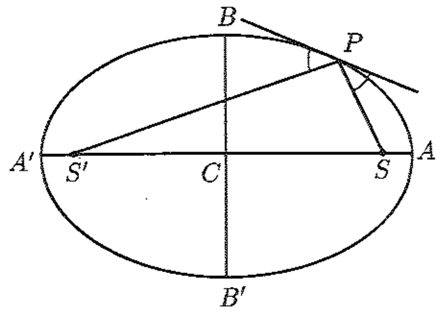

Figure A.4

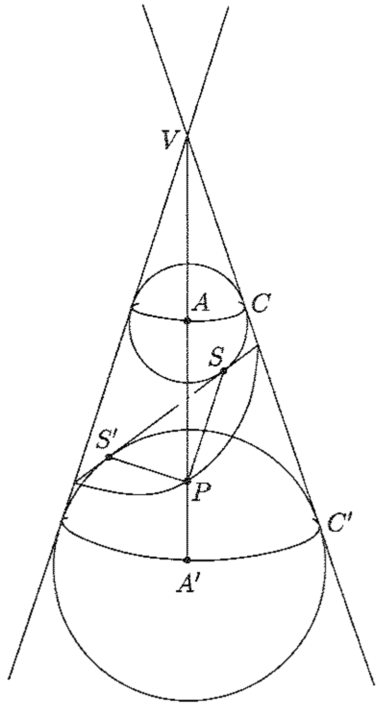

Figure A.5

any point on the ellipse, and let the line VP through the vertex of the cone meet C and C' in points A and A', respectively. PS and PA are lines tangent to the upper sphere at S and A, so $PS = PA$. Similarly, $PS' = PA'$. So $PS + PS' = PA + PA'$, which is independent of the position of P.

The lines SP and $S'P$ make equal angles with the tangent at P. See Figure A.5. Let s be arc length measured along the curve, and let $SP = r$, $SP' = r'$; then from the property just noted,

$$\frac{dr}{ds} + \frac{dr'}{ds} = 0.$$

But dr/ds is equal to the cosine of the angle between the line SP and the tangent, with a similar result for dr'/ds. The result follows at once.

From the form of the cartesian equation of the ellipse, (A.2.1), it follows that the ellipse is the orthogonal projection of a circle of radius a; this is the *auxiliary circle*. Alternatively, draw a circle of radius a, and choose any diameter AA'. From any point Q on the circumference, drop a perpendicular QR onto AA', and construct P on QR such that

$$\frac{PR}{QR} = \frac{b}{a}.$$

Then P will trace out the ellipse with semiaxes a and b. (See Figure A.6.)

It is easily verified from this property that the area of the ellipse is

$$\pi ab.$$

Let $\angle QCA = E$; then the cartesian coordinates of P are

$$x = a\cos E, \quad y = b\sin E. \tag{A.2.3}$$

E is called the *eccentric angle* of P.

A.3 The Parabola

The conic with $e = 1$ is the parabola. If we let e tend to one for an ellipse, then a tends to infinity. Although some properties of the parabola can be deduced from corresponding properties of the ellipse by letting $e = 1$ and $1/a = 0$, this may lead to expressions that are meaningless; as, for instance, if it is applied to (A.2.2). However, if we remember that the pericenter distance, $q = SA$, remains finite, and write (A.2.2) as

$$\frac{q(1+e)}{r} = 1 + e\cos v,$$

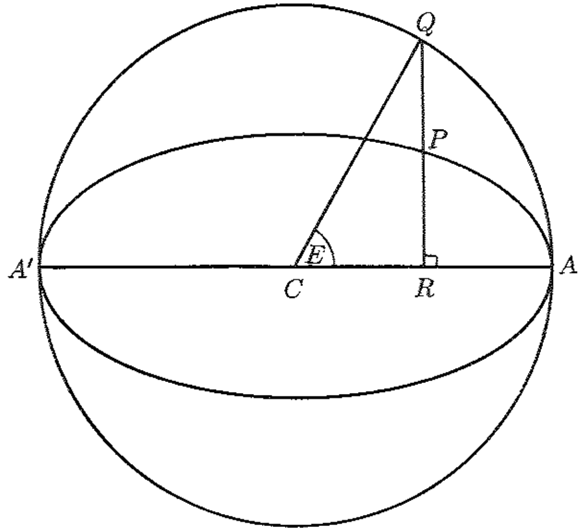

Figure A.6

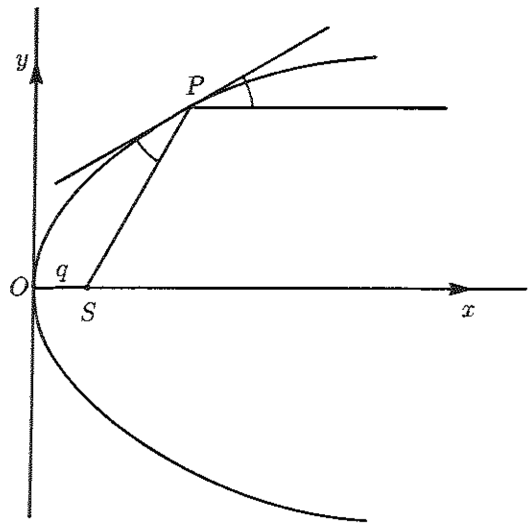

Figure A.7

then there is no difficulty in putting $e = 1$, and we find the polar equation of the parabola in the usual form:

$$r = q \sec^2 \frac{v}{2}. \tag{A.3.1}$$

The size of a parabola is usually given by q. (All parabolas have the same shape.) The semilatus rectum is $2q$.

The cartesian equation can be put into the form

$$y^2 = 4qx. \tag{A.3.2}$$

There is only one focus, at $(q, 0)$, and one axis, the x-axis.

Return for a moment to the ellipse, and let e tend to one. The second focus tends to infinity, and the theorem that the tangent at P makes equal angles with SP and $S'P$ becomes, for the parabola, that the tangent at P makes equal angles with SP and the axis of the parabola. This leads to the well-known property that a ray of light traveling parallel to the axis of a parabolic mirror passes through the focus after reflection. (See Figure A.7.)

Sometimes it is convenient to consider a curve as traced out by the variation of a single variable, or *parameter*; the equations (A.2.3) are an example of this; they are one form (out of many possibilities) of the *parametric equations* of the ellipse. There are many possible representations of the parabola; perhaps the simplest is

$$x = qt^2, y = 2qt. \tag{A.3.3}$$

A.4 The Hyperbola

The hyperbola is defined by $e > 1$. From the polar equation we see that the curve is not bounded, since r can become arbitrarily large. When r does become large, v tends to one of the values given by

$$v_a = \cos^{-1}\left(-\frac{1}{e}\right). \tag{A.4.1}$$

The cartesian equation can be put into the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{A.4.2}$$

In this section, for ease of geometrical description, a and b will be taken to be positive. But note that in computation, and in formulas for Keplerian motion, a should be a negative number. When x and y become large, (A.4.2) is nearly the same as

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0, \tag{A.4.3}$$

which is the equation of two lines through the origin. The larger r becomes, the more nearly the curve resembles these two lines; they are called *asymptotes*.

The directions given by the values of v_a, from (A.4.1), must be parallel to these asymptotes, and we deduce that the angle between the asymptotes is

$$2\cos^{-1}\left(-\frac{1}{e}\right) \quad \text{or} \quad \cos^{-1}\left(\frac{2}{e^2} - 1\right).$$

From (A.4.2) we see that there is no point on the hyperbola for which $-a < x < a$; therefore the hyperbola has two branches; the polar equation gives only one of these. (See Figure A.8.) Comparing the gradients of the asymptotes, (A.4.3), with the formula v_a, we find

$$\tan^{-1}\frac{b}{a} = \cos^{-1}\frac{1}{e}$$

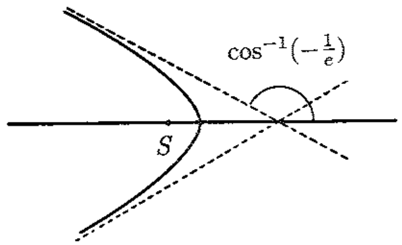

Figure A.8

from which we deduce

$$b^2 = a^2(e^2 - 1).$$

b can be less or greater than a; Figures A.9(a), (b), and (c) show hyperbolas for which b is greater than, equal to, and less than a. If $b = a$, then the figure is a *rectangular hyperbola*.

In the notation of Figure A.10, we see that

$$CA = a,$$
$$AS = q = a(e - 1).$$

The tangent at A cuts the asymptotes in points $(-a, \pm b)$. The semilatus rectum is

$$p = a(e^2 - 1).$$

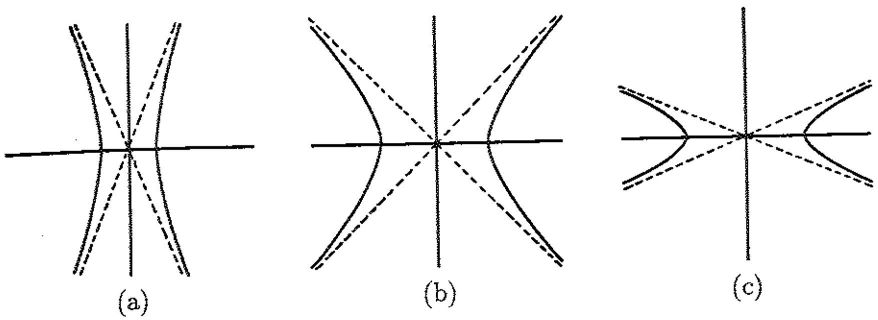

Figure A.9

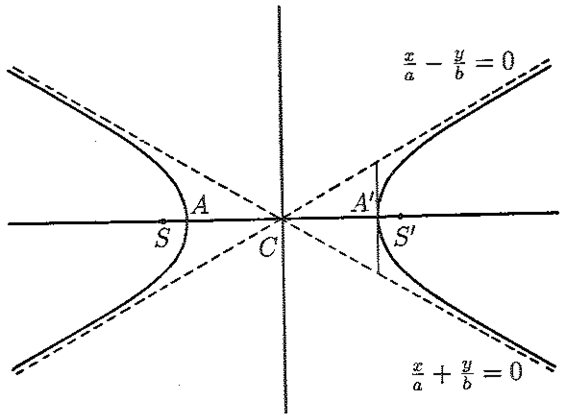

Figure A.10

The hyperbola can be defined as the locus of P such that the difference between SP and $S'P$ is constant, S and S' being fixed points. These are the foci, and to recover the equations given above, the constant difference must be equal to $2a$, so that

$$SS' = 2c = 2ae.$$

The cartesian coordinates of any point on the hyperbola with equation (A.4.2) are given by the parametric equations

$$x = \pm a \cosh F, \quad y = b \sinh F. \tag{A.4.4}$$

A.5 Pole and Polar

This section is required for the interpretation of Hamilton's theorem, Section 4.9.

Let $P(x_1, y_1)$ be any point from which real tangents can be drawn to a conic with the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{A.5.1}$$

Let these tangents meet the conic at points Q and R; then it is not difficult to establish that the equation of the line QR is

$$x(ax_1 + hy_1 + g) + y(by_1 + hx_1 + f) + gx_1 + fy_1 + c = 0. \tag{A.5.2}$$

This line is the polar of P with respect to the conic. Even if real tangents cannot be drawn from P, (A.5.2) still defines the polar of P. Conversely, P is the pole of the line with equation (A.5.2).

The reader will be able to verify without difficulty that the focus and directrix are pole and polar.

If P lies on the conic, then (A.5.2) is the equation of the tangent at P.

Appendix B

The Rotation of Axes

If a system of right-handed rectangular axes is rotated positively through θ about the x-axis, the new coordinates, (x', y', z'), of a point are related to the old, (x, y, z), by

$$\begin{aligned} x' &= x, \\ y' &= y\cos\theta + z\sin\theta, \\ z' &= -y\sin\theta + z\cos\theta. \end{aligned}$$

These equations can be written in matrix notation as

$$\begin{aligned} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= P(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \end{aligned}$$

where

$$P(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

Similarly, a positive rotation through θ about the y-axis leads to new coordinates given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = Q(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where

$$Q(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

And a positive rotation through θ about the z-axis leads to new coordinates given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For a combination of two rotations, let us say $R(\theta)$ followed by $P(\phi)$, (the order is crucially important), we have

$$\begin{bmatrix} x' \\ y' \\ x' \end{bmatrix} = P(\phi)\ R(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\cos\phi\sin\theta & \cos\phi\cos\theta & \sin\phi \\ \sin\phi\sin\theta & -\sin\phi\cos\theta & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

In the same way the matrix describing any sequence of rotations can be found by straightforward matrix multiplication. For example, suppose that (X, Y, Z) are coordinates of a comet in the orbital reference system, with OX pointing toward perihelion, and OZ parallel to $\mathbf{h}$, (so that $Z = 0$), and that (x, y, z) are the coordinates of the comet in an equatorial system of axes with Ox pointing toward the vernal equinox and Oz toward the north celestial pole. Then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P(-\epsilon)\ \ R(-\Omega)\ \ P(-i)\ \ R(-\omega) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

Be very careful with signs of rotations. The transformations apply equally to velocity components. They can be used to advantage with "infinitesimal" rotations. In fact their use, and the use of vector operations, almost completely obviates the need for spherical trigonometry.

Appendix C

Numerical Values

Numerical values continue to be refined, and further objects will be found that might be included in tables such as these. Don't be surprised if quite different values appear in other references (and even within the same reference). In some instances there are "official" values, so that different computations share a common set, making comparison possible. The following values are included partly for their interest, and partly so that you can include them, if you wish, in your computations.

C.1 Orbital Elements of Planets

The data for the planets (excluding Pluto) were kindly supplied by L. E. Doggett from data of P. Bretagnon (Reference 72). They are mean elements, referred to the equinox J2000.0. T is measured in Julian centuries from that epoch. The semimajor axes have been derived from the terms in L (the mean longitude at epoch) factored by T, with allowance first made for the precession in longitude. (See Appendix C7.)

For detailed information on osculating elements, refer to the *Astronomical Almanac*. For low-precision formulas, see Reference 38.

Mercury ☿

$$\tilde{\omega} = 77.4561\ 19 + 1.5564\ 77 \times T + 0.0002\ 96 \times T^2$$
$$\Omega = 48.3308\ 93 + 1.1861\ 88 \times T + 0.0001\ 76 \times T^2$$
$$i = 7.0049\ 86 + 0.0018\ 21 \times T - 0.0000\ 18 \times T^2$$
$$e = 0.2056\ 3175 + 0.0000\ 2041 \times T - 0.0000\ 0003 \times T^2$$
$$L = 252.2509\ 06 + 149474.0722\ 49 \times T + 0.0003\ 04 \times T^2$$
$$a = 0.3871\ 0353$$

Venus ♀

$$\tilde{\omega} = 131.5637\ 07 + 1.4022\ 21 \times T - 0.0010\ 73 \times T^2 - 0.0000\ 05 \times T^3$$
$$\Omega = 76.6799\ 20 + 0.9011\ 20 \times T + 0.0004\ 07 \times T^2$$
$$i = 3.3946\ 62 + 0.0010\ 04 \times T - 0.0000\ 01 \times T^2$$
$$e = 0.0067\ 7188 - 0.0000\ 4777 \times T$$
$$L = 181.9798\ 01 + 58519.2130\ 30 \times T + 0.0003\ 11 \times T^2$$
$$a = 0.7233\ 074$$

Earth ♁

$$\tilde{\omega} = 102.9373\ 48 + 1.7195\ 39 \times T + 0.0004\ 60 \times T^2$$
$$e = 0.0167\ 0862 - 0.0000\ 4204 \times T$$
$$L = 100.4664\ 49 + 36000.7698\ 23 \times T + 0.0003\ 04 \times T^2$$
$$a = 1.0000\ 1161$$

Mars ♂

$$\tilde{\omega} = 336.0602\ 34 + 1.8410\ 44 \times T + 0.0001\ 35 \times T^2$$
$$\Omega = 49.5580\ 93 + 0.7720\ 96 \times T + 0.0000\ 16 \times T^2$$
$$i = 1.8497\ 26 - 0.0006\ 01 \times T + 0.0000\ 13 \times T^2$$
$$e = 0.0934\ 0062 + 0.0000\ 9048 \times T - 0.0000\ 0008 \times T^2$$
$$L = 355.4332\ 75 + 19141.6964\ 75 \times T + 0.0003\ 11 \times T^2$$
$$a = 1.5237\ 1069$$

Jupiter ♃

$$\tilde{\omega} = 374.3313\ 09 + 1.6126\ 38 \times T + 0.0010\ 31 \times T^2 - 0.0000\ 04 \times T^3$$
$$\Omega = 100.4644\ 41 + 1.0209\ 54 \times T + 0.0004\ 01 \times T^2$$
$$i = 1.3032\ 70 - 0.0054\ 97 \times T + 0.0000\ 05 \times T^2$$
$$e = 0.0484\ 9485 + 0.0001\ 6324 \times T - 0.0000\ 0047 \times T^2$$
$$L = 34.3514\ 84 + 3036.3027\ 89 \times T + 0.0002\ 24 \times T^2$$
$$a = 5.2102\ 1558$$

Saturn ♄

$$\tilde{\omega} = 93.0567\ 87 + 1.9637\ 61 \times T + 0.0008\ 38 \times T^2 + 0.0000\ 05 \times T^3$$
$$\Omega = 113.6655\ 24 + 0.8770\ 94 \times T - 0.0001\ 21 \times T^2 - 0.0000\ 02 \times T^3$$
$$i = 2.4888\ 78 - 0.0037\ 36 \times T - 0.0000\ 15 \times T^2$$
$$e = 0.0555\ 0862 - 0.0003\ 4682 \times T - 0.0000\ 01 \times T^2$$
$$L = 50.0774\ 71 + 1223.5110\ 14 \times T + 0.0005\ 20 \times T^2$$
$$a = 9.5380\ 7012$$

Uranus ♅

$$\tilde{\omega} = 173.0051\ 59 + 1.4863\ 79 \times T + 0.0002\ 15 \times T^2$$
$$\Omega = 74.0059\ 47 + 0.5211\ 27 \times T + 0.0013\ 40 \times T^2 + 0.0000\ 19 \times T^3$$
$$i = 0.7731\ 96 + 0.0007\ 74 \times T + 0.0000\ 37 \times T^2$$
$$e = 0.0462\ 9590 - 0.0000\ 2734 \times T + 0.0000\ 0008 \times T^3$$
$$L = 314.0550\ 05 + 429.8640\ 56 \times T + 0.0003\ 04 \times T^2$$
$$a = 19.1833\ 02$$

Neptune ♆

$$\tilde{\omega} = 48.1236\ 91 + 1.4262\ 70 \times T + 0.0003\ 79 \times T^2$$
$$\Omega = 131.7840\ 57 + 1.1022\ 03 \times T + 0.0002\ 60 \times T^2$$
$$i = 1.7699\ 52 - 0.0093\ 08 \times T - 0.0000\ 08 \times T^2$$
$$e = 0.0089\ 8809 + 0.0000\ 0641 \times T$$
$$L = 304.3486\ 65 + 219.8833\ 09 \times T + 0.0003\ 09 \times T^2$$
$$a = 30.0551\ 44$$

The following elements for Pluto are taken from the *Astronomical Almanac* of 1986. They are for 1986.0.

Pluto ♇

$$\tilde{\omega} = 224.6148$$
$$\Omega = 110.4065$$
$$i = 17.1323\ 3$$
$$e = 0.2508\ 77$$
$$L = 218.8873\ 5$$
$$a = 39.5375\ 8$$

C.2 Satellites — Orbital and Physical Data

Satellites — Orbital and Physical Data, Part I

Planet and Satellite	Orbital Period[1] R = retrograde days	Semimajor axis 10^3km	Orbital eccentricity
Earth			
Moon	27.321 661	384.400	0.054 900 489
Mars			
I Phobus	0.318 910 23	9.378	0.015
II Deimos	1.262 440 7	23.459	0.000 5
Jupiter			
I Io	1.769 137 786	422	0.004
II Europa	3.551 181 041	671	0.009
III Ganymede	7.154 552 96	1070	0.002
IV Callisto	16.689 018 4	1883	0.007
V Amalthea	0.498 179 05	181	0.003
VI Himalia	250.566 2	11480	0.157 98
VII Elara	259.652 8	11737	0.207 19
VIII Pasiphae	735R	23500	0.378
IX Sinope	758R	23700	0.275
X Lysithea	259.22	11720	0.107
XI Carme	692R	22600	0.206 78
XII Anake	631R	21200	0.168 70
XIII Leda	238.72	11094	0.147 62
XIV Thebe	0.674 5	222	0.015
XV Adrastea	0.298 26	129	
XVI Metis	0.294 780	128	
Saturn			
I Mimas	0.942 421 813	185.52	0.020 2
II Enceladus	1.370 217 855	238.02	0.004 52
III Tethys	1.887 802 160	294.66	0.000 00
IV Dione	2.736 914 742	377.40	0.002 230
V Rhea	4.517 500 436	527.04	0.001 00
VI Titan	15.945 420 68	1221.83	0.029 192
VII Hyperion	21.276 608 8	1481.1	0.104
VIII Iapetus	79.330 182 5	3561.3	0.028 28
IX Phoebe	550.48 R	12952	0.163 26
X Janus	0.694 5	151.472	0.007
XI Epimethus	0.694 2	151.422	0.009
XII 1980S6	2.736 9	377.40	0.005
XIII Telesto	1.887 8	294.66	
XIV Calypso	1.887 8	294.66	
XV Atlas	0.601 9	137.670	0.000
1980S26	0.628 5	141.700	0.004
1980S27	0.613 0	139.353	0.003
Uranus			
I Ariel	2.520 379 35	191.02	0.0034
II Umbriel	4.144 177 2	266.30	0.005 0
III Titania	8.705 871 7	435.91	0.002 2
IV Oberon	13.463 238 9	583.52	0.000 8
V Miranda	1.413 479 25	129.39	0.002 7
Neptune			
I Triton	5.876 843 3 R	354.29	< 0.01
II Nereid	360.2	5511	0.748 3
Pluto			

Satellites — Orbital and Physical Data, Part II

Orbital Inclination to Planetary Equator degrees	Sidereal Period of Rotation S = synchronous days	Radius km	Mass 1/Planet	Planet
				Earth
18.28–28.58	S	1738	0.0123 000 2	Moon
				Mars
1.0	S	$13.5 \times 10.8 \times 9.4$	1.5×10^{-8}	I Phobus
0.9–2.7	S	$7.5 \times 6.1 \times 5.5$	3×10^{-9}	II Deimos
				Jupiter
0.04	S	1815	4.68×10^{-5}	I Io
0.47	S	1569	2.52×10^{-5}	II Europa
0.21	S	2631	7.80×10^{-5}	III Ganymede
0.51	S	2400	5.66×10^{-5}	IV Callisto
0.40	S	$135 \times 83 \times 75$	38×10^{-10}	V Amalthea
27.63	0.4	93	50×10^{-10}	VI Himalia
24.77	0.5	38	4×10^{-10}	VII Elara
145		25	1×10^{-10}	VIII Pasiphae
153		18	0.4×10^{-10}	IX Sinope
29.02		18	0.4×10^{-10}	X Lysithea
164		20	0.5×10^{-10}	XI Carme
147		15	0.2×10^{-10}	XII Anake
26.07		8	0.3×10^{-11}	XIII Leda
0.08		55×45	4×10^{-10}	XIV Thebe
		$12.5 \times 10 \times 7.5$	0.1×10^{-10}	XV Adrastea
		20	0.5×10^{-10}	XVI Metis
				Saturn
1.53	S	198	8.0×10^{-8}	I Mimas
0.00	S	253	1.3×10^{-7}	II Enceladus
1.86	S	525	1.3×10^{-6}	III Tethys
0.02	S	560	1.85×10^{-6}	IV Dione
0.35	S	765	4.4×10^{-6}	V Rhea
0.33	S	2575	2.38×10^{-4}	VI Titan
0.43		$205 \times 130 \times 110$	3×10^{-8}	VII Hyperion
14.72	S	718	3.3×10^{-6}	VIII Iapetus
177[2]	0.4	110	7×10^{-10}	IX Phoebe
0.14	S	$110 \times 100 \times 80$		X Janus
0.34	S	$70 \times 60 \times 50$		XI Epimethus
0.0		$18 \times 16 \times 15$		XII 1980S6
		$17 \times 14 \times 13$		XIII Telesto
		$17 \times 11 \times 11$		XIV Calypso
0.3		20×10		XV Atlas
0.0		$55 \times 45 \times 35$		1980S26
0.0		$70 \times 50 \times 40$		1980S27
				Uranus
0.3		580	1.8×10^{-5}	I Ariel
0.36		595	1.2×10^{-5}	II Umbriel
0.14		800	6.8×10^{-5}	III Titania
0.10	S	775	6.9×10^{-5}	IV Oberon
4.2		240	0.2×10^{-5}	V Miranda
				Neptune
159.90	S	1600	1.3×10^{-3}	I Triton
27.6[3]		1500	2×10^{-7}	II Nereid
				Pluto

C.3 Physical Elements of Planets

Most of these data are taken, with permission, from the *Astronomical Almanac.* Figures for the equatorial radii and flattening are those recommended in a report of the IAU/IAG/COSPAR working group on cartographic coordinates and rotational elements of the planets and satellites, published in 1985. The quantity "flattening" is defined as:

$$\frac{(\text{equatorial radius}) - (\text{polar radius})}{\text{equatorial radius}}.$$

Planet	Equatorial Radius km	Flattening	Mass 10^{24}kg	Mean Density gm/cm^3
Mercury	2439	0	0.330 22	5.43
Venus	6051	0	4.869 0	5.24
Earth	6378.140	0.003 352 81	5.974 2	5.515
Mars	3393.4	0.005 186 5	0.641 91	3.94
Jupiter	71398	0.064 808 8	1898.8	1.33
Saturn	60000	0.107 620 9	568.50	0.70
Uranus	25400	0.030	86.625	1.30
Neptune	25295	0.022	102.78	1.76
Pluto	1500	0	0.015?	?

Planet	Period of rotation days	Inclination of equator to plane of orbit degrees	Velocity of escape km/sec	Acceleration due to gravity at the equator cm/sec^2
Mercury	58.646 2	0.0	4.2	370
Venus	243.01	177.3	10.4	886
Earth	0.997 269 68	23.45	11.2	979.2
Mars	1.025 956 75	25.19	5.0	384
Jupiter	0.413 54*	3.12	60	2500
Saturn	0.437 5	26.73	36	1050
Uranus	0.65	97.86	21	900
Neptune	0.768	29.56	24	1160
Pluto	6.386 7	118?	?	?

Planet	Centrifugal acceleration at the equator cm/sec^2	J_2	J_3	J_4
Mercury	-0.00	—	—	—
Venus	-0.00	—	—	—
Earth	-3.39	$+1.082\ 63 \times 10^{-3}$	-0.254×10^{-5}	-0.161×10^{-5}
Mars	-1.70	$+1.964 \times 10^{-3}$	$+0.36 \times 10^{-4}$	—
Jupiter	-221	$+14.75 \times 10^{-3}$	—	-0.58×10^{-3}
Saturn	-166	16.45×10^{-3}	—	-0.10×10^{-2}
Uranus	-32	$+12 \times 10^{-3}$	—	—
Neptune	-22	$+4 \times 10^{-3}$	—	—
Pluto	?	—	—	—

*The period of rotation becomes longer toward the poles.

Dimensions of the Principal Rings of Saturn

Radius (limiting values quoted)	km
	138000
Outer A ring: moderately bright	
	120000
Cassini division: dark	
	116000
Main B ring: very bright	
	90000
Gap: dark	
	89000
Crape or C ring: faint	
	71000

These are, in fact, subdivided into many narrower rings; more rings have been observed.

Dimensions of Rings of Uranus

Ring	Semimajor axis (km)	Eccentricity
6	41870	0.0014
5	42270	0.0018
4	42600	0.0012
α	44750	0.0007
β	45700	0.0005
η	47210	—
γ	47660	—
δ	48330	0.0005
ε	51180	0.0079

A ring of Jupiter has been observed; the outer edge has radius 1.81 times the radius of Jupiter and the ring is approximately 7000 km wide.

C.4 The Earth

Lengths of the Principal Years (1986.0)

Tropical	$365^{d}.242\ 191$
Sidereal	365.256 363
Anomalistic	365.249 635
Eclipse	346.620 071
Julian	365.25

Length of the Day

Mean solar day	$24^h03^m56^s.555$ sidereal time
or	1.002 737 91 sidereal days.
Sidereal day	$23^h56^m4^s.091$ mean solar time
or	0.997 269 57 mean solar days.

Dimensions

Equatorial radius	$a = 6378.140$ km
Polar radius	$c = 6356.755$ km
Mass	$M = 5.9742 \times 10^{24}$ kg

Moments of Inertia

About the axis of rotation, $C = 8.01 \times 10^{44}$ gm cm^2
$= 0.330$ Ma^2.
About an equatorial axis, $A = 0.329$ Ma^2.

C.5 The Moon

Lengths of the Principal Months (1986.0)

Sidereal month	$27^d.321\,662$
Synodic month	29.530 589
Anomalistic month	27.554 550
Nodical month	27.212 221
Tropical month	27.321 582

Orbit

Mean distance from the Earth	384400 km
or	0.002570 AU
Inclination to the ecliptic	5°9′
Inclination of lunar equator to the ecliptic	1°32′
Eccentricity	0.05490
Period of the node	18.60 tropical years

Dimensions

Radius	1738.0 km
Mass	7.3483×10^{22} kg
or	(mass of the Earth)/81.301
Mean density	3.94 gm/cm^3
Surface gravity	162.2 cm/sec^2
Escape speed	2.37 km/sec
Moment of inertia about rotation axis	C = 0.392 MR^2
Moment of inertia differences	$(B - A)/C = 0.000\,2278$, $(C - A)/B = 0.000\,6313$.

C.6 The Sun

Radius	696265 km
Mass	1.9981×10^{30} kg
Period of rotation at equator	$25^d.38$
(The period of rotation increases toward the poles.)	
Inclination of the solar equator to the ecliptic	7°.25
Solar parallax	8″.794 148
Mean distance to the Earth (astronomical unit)	$1.495\,978\,70 \times 10^{11}$ m

C.7 Physical Constants

Speed of light	c = 299 792 458 m/s
Constant of gravitation	$G = 6.672 \times 10^{-11}\, m^3$ kg^{-1} s^{-2}
Gaussian gravitational constant	k = 0.017 202 09895
Geocentric gravitational constant	$3.986\,005 \times 10^{14}\, m^3\, s^{-2}$
Obliquity of the ecliptic at epoch 2000	23°26′21″.448
Constant of nutation at epoch 2000	9″.2025
Constant of precession at epoch 2000	5029″.0966
Constant of aberration	20″.47
Number of ephemeris seconds in one tropical year (1900)	31 556 925.9747

C.8 Miscellaneous Data

1 foot	= 30.4800 cm
1 mile	= 1.609 344 km
1 nautical mile	= 6080 ft = 1.853 km
Seconds in a day	= 86 400
Seconds in a tropical year	= 3.155 6026 × 10^7
1 mile/hour	= 44.704 cm/sec
	= 1.4667 ft/sec
e	= 2.7182 8182 8459 0452 3536...
π	= 3.1415 9265 3589 7932 3846...
1 radian	= 57°.295 7795...
1°	= 0.017 453 2925... radian

Appendix D

MISCELLANEOUS EXPANSIONS IN SERIES

D.1 f and g Series

$$\begin{aligned} f &= 1 - \tfrac{1}{2}\sigma t^2 + \tfrac{1}{2}\sigma\tau t^3 + \tfrac{1}{24}\sigma(3\omega - 2\sigma - 15\tau^2)t^4 - \tfrac{1}{8}\sigma\tau(3\omega - 2\sigma - 7\tau^2)t^5 \\ &\quad + \tfrac{1}{720}\sigma\{(630\omega - 420\sigma - 945\tau^2)\tau^2 - (22\sigma^2 - 66\sigma\omega + 45\omega^2)\}t^6 + \cdots \\ g &= t - \tfrac{1}{6}\sigma t^3 + \tfrac{1}{4}\sigma\tau t^4 + \tfrac{1}{120}\sigma(9\omega - 8\sigma - 45\tau^2)t^5 \\ &\quad - \tfrac{1}{24}\sigma\tau(6\omega - 5\sigma - 14\tau^2)t^6 + \cdots \end{aligned}$$

where

$$\sigma = \frac{1}{r_0^3}, \quad \tau = \frac{\dot{r}_0}{r_0}, \quad \omega = \frac{\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_0}{r_0^2}.$$

D.2 Elliptic Motion

$$\begin{aligned} E &= M + 2\sum_{n=1}^{\infty} \frac{1}{n} J_n(ne) \sin nM \\ \cos E &= -\frac{1}{2}e + 2\sum_{n=1}^{\infty} \frac{1}{n} J_n'(ne) \cos nM \\ \sin E &= \frac{2}{e}\sum_{n=1}^{\infty} \frac{1}{n} J_n'(ne) \sin nM \\ \sin v &= 2\sqrt{1-e^2}\sum_{n=1}^{\infty} J_n(ne) \sin nM \\ \cos v &= -e + \frac{2(1-e^2)}{e}\sum_{n=1}^{\infty} J_n(ne) \cos nM \end{aligned}$$

$$v - M = (2e - \tfrac{1}{4}e^3)\sin M + (\tfrac{5}{4}e^2 - \tfrac{11}{24}e^4)\sin 2M + \tfrac{13}{12}e^3\sin 3M + \tfrac{103}{96}e^4\sin 4M + \cdots$$

The J_n and J'_n are Bessel functions. The first few are given by

$$\begin{aligned}
J_1(e) &= \tfrac{1}{2}e(1 - \tfrac{1}{8}e^2 + \tfrac{1}{192}e^4 - \tfrac{1}{9216}e^6 + \cdots),\\
J_2(2e) &= \tfrac{1}{2}e^2(1 - \tfrac{1}{3}e^2 + \tfrac{1}{24}e^4 - \tfrac{1}{360}e^6 + \cdots),\\
J_3(3e) &= \tfrac{9}{16}e^3(1 - \tfrac{9}{16}e^2 + \tfrac{81}{640}e^4 - \cdots),\\
J_4(4e) &= \tfrac{2}{3}e^4(1 - \tfrac{4}{5}e^2 + \tfrac{4}{15}e^4 - \cdots),\\
J_5(5e) &= \tfrac{625}{768}e^5(1 - \tfrac{25}{24}e^2 + \tfrac{625}{1344}e^4 - \cdots),\\
J_6(6e) &= \tfrac{81}{80}e^6(1 - \tfrac{9}{7}e^2 + \tfrac{81}{112}e^4 - \cdots),
\end{aligned}$$

and

$$\begin{aligned}
J'_1(e) &= \tfrac{1}{2}(1 - \tfrac{3}{8}e^2 + \tfrac{5}{192}e^4 - \tfrac{7}{9216}e^6 + \cdots),\\
J'_2(2e) &= \tfrac{1}{2}e(1 - \tfrac{2}{3}e^2 + \tfrac{1}{8}e^2 - \tfrac{1}{90}e^6 + \cdots),\\
J'_3(3e) &= \tfrac{9}{16}e^3(1 - \tfrac{15}{16}e^2 + \tfrac{189}{640}e^4 - \cdots),\\
J'_4(4e) &= \tfrac{2}{3}e^3(1 - \tfrac{6}{5}e^2 + \tfrac{8}{15}e^4 - \cdots),\\
J'_5(5e) &= \tfrac{625}{768}e^4(1 - \tfrac{35}{24}e^2 + \tfrac{375}{448}e^4 - \cdots),\\
J'_6(6e) &= \tfrac{81}{80}e^5(1 - \tfrac{12}{7}e^2 + \tfrac{135}{112}e^4 - \cdots).
\end{aligned}$$

In general,

$$J_n(ne) = \frac{e}{2}\left(\frac{ne}{2}\right)^{n-1}\frac{1}{(n-1)!}\left\{1 - \frac{n^2e^2}{2\cdot(2n+2)} + \frac{n^4e^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \cdots\right\},$$

and

$$J'_n(ne) = \frac{1}{2}\left(\frac{ne}{2}\right)^{n-1}\frac{1}{(n-1)!}\left\{1 - \frac{n+2}{n}\frac{n^2e^2}{2(2n+2)} + \frac{n+4}{n}\frac{n^4e^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \cdots\right\}.$$

The power series for the Bessel functions converge for all values of e. But if the series for quantities in elliptic motion are expressed explicitly as power series in e, then they converge for $|e| < 0.6627434\ldots$ See Ref. 26.

Appendix E

The Solution of Linear Systems

Two programs are listed here. One is for solving the system $Ax = b$, assuming n equations for n unknowns, and the other is for inverting the matrix A. These are required particularly in connection with the improvement of orbits. The programs use Gaussian elimination with partial pivoting. Here the elimination takes place starting with the leftmost column and moving systematically to the right, but the row that contains the pivot is chosen by the program. First, the largest number (in absolute value) in each row of A is found, and this provides a *weight* for the row. Each possible candidate for a pivot is divided by the weight for its row; the pivot chosen is that with the largest resulting number. The program for inversion is wasteful of storage space, but is, I hope, clear. The final product of the pivots gives the value of the determinant of A. If at any stage of the program the product of the pivots used so far becomes less (in absolute value) than a small number, which the operator provides, then the system is declared to be ill-conditioned, and execution stops.

```
10 CLS
20 PRINT "   THE PROGRAM SOLVES A LINEAR SYSTEM OF N EQUATIONS."
30 PRINT "N CAN BE AS LARGE AS 6, OR GREATER IF THE DIMENSIONS"
40 PRINT "ARE CHANGED."
50 PRINT "   THE SYSTEM IS AX = B, WITH X THE VECTOR OF"
60 PRINT "UNKNOWNS. THE COMPONENTS OF A AND B SHOULD BE
70 PRINT "ENTERED BY COLUMN INTO A DATA STATEMENT.  IN THE"
80 PRINT "PROGRAM, B(K) IS LOCATED IN A(K,N+1)."
90 PRINT "   THE PROGRAM USES GAUSSIAN ELIMINATION WITH PARTIAL"
100 PRINT "PIVOTING. IT ALSO CALCULATES THE ABSOLUTE VALUE OF"
110 PRINT "THE DETERMINANT.  IF AT ANY STAGE THE ABSOLUTE VALUE"
120 PRINT "OF THE PRODUCT OF THE PIVOTS IS SMALLER THAN A NUMBER"
130 PRINT "EPS, TO BE ENTERED BY THE OPERATOR, THE PROGRAM WILL"
140 PRINT "END." : PRINT
150 PRINT "    PRESS A KEY TO CONTINUE."
160 A$ = INKEY$ : IF A$ = "" THEN 160
170 PRINT : PRINT "   NOW ENTER THE DATA AS:" : PRINT : PRINT
180 PRINT "1000 DATA N, EPS, A(1,1), A(2,1),..., A(N,1),"
190 PRINT "A(1,2),..., A(N,2), A(1,3),..., A(N-1,N), A(N,N),"
200 PRINT "B(1),..., B(N)"
210 PRINT "USE EXTRA DATA STATEMENT LINES AS NECESSARY."
```

```
220 PRINT : PRINT
230 PRINT "THEN ENTER 'GOTO 250', AND HIT 'RETURN'." : PRINT
240 END
250 DEFDBL A-H,O-Z : DEFINT I-N
260 DIM A(6,7), W(6), X(6), L(6)
270     REM L IS A VECTOR THAT KEEPS TRACK OF THE
280     REM LOCATIONS OF THE PIVOTS.
290 READ N, EPS
300 D = 1# : REM D WILL EVENTUALLY BECOME THE DETERMINANT
310 FOR J = 1 TO N + 1
320    FOR I = 1 TO N
330       READ A(I,J)
340    NEXT I
350 NEXT J
360     REM IN LINES 370 TO 440 THE WEIGHTS, W(I), ARE FOUND.
370 FOR I = 1 TO N
380    L(I) = I
390    W(I) = ABS(A(I,1))
400    FOR J = 2 TO N
410       IF W(I) > ABS(A(I,J)) THEN 430
420       W(I) = ABS(A(I,J))
430    NEXT J
440 NEXT I
450     REM NOW START THE LARGE LOOP FOR SUCCESSIVE ELIMINATIONS.
460 FOR K = 1 TO N - 1
470     REM SELECT AND IDENTIFY THE PIVOT.
480     REM J DEFINES THE PIVOT ROW.
490    RM = 0#
500    FOR I = K TO N
510       R = ABS(A(L(I),K))/W(L(I))
520       IF R < RM THEN 550
530       J = I
540       RM = R
550    NEXT I
560    LT = L(J) : L(J) = L(K) : L(K) = LT
570    D = D*A(LT,K)
580    IF ABS(D)<EPS THEN 830
590     REM PERFORM THE ELIMINATION.
600    FOR I = K + 1 TO N
610       PF = A(L(I),K)/A(LT,K)
620       FOR J = K + 1 TO N + 1
630          A(L(I),J) = A(L(I),J) - PF*A(LT,J)
640       NEXT J
650    NEXT I
660 NEXT K
670 D = D*A(L(N),N)
680 PRINT "THE ABSOLUTE VALUE OF THE DETERMINANT IS " ; ABS(D)
690 IF ABS(D)<EPS THEN 830
700     REM BACK SUBSTITUTION.
710 X(N) = A(L(N),N + 1)/A(L(N),N)
720 FOR I = 1 TO N - 1
730    XT = A(L(N - I),N + 1)
740    FOR J = N - I + 1 TO N
750       XT = XT - A(L(N - I),J)*X(L(J))
760    NEXT J
770    X(L(N - I)) = XT/A(L(N - I),N - I)
780 NEXT I
```



```
790 FOR I = 1 TO N
800    PRINT "X("; I ;") = " ; X(L(I))
810 NEXT I
820 END
830 PRINT "THE SYSTEM IS ILL-CONDITIONED, AND EXECUTION"
840 PRINT "HAS ENDED."
850 END
1000 DATA 3,0.001,1,1,2,1,-1,1,1,1,-1,3,1,3

10 CLS
20 PRINT "   THIS PROGRAM INVERTS AN N-BY-N MATRIX A. N CAN
30 PRINT "BE AS LARGE AS 6, OR GREATER, IF THE DIMENSIONS"
40 PRINT "CHANGED. THE INVERSE MATRIX IS B." : PRINT
50 PRINT "   THE PROGRAM USES GAUSSIAN ELIMINATION WITH
60 PRINT "PARTIAL PIVOTING. IT ALSO CALCULATES THE ABSOLUTE"
70 PRINT "VALUE OF THE DETERMINANT. IF, AT ANY STAGE THE"
80 PRINT "ABSOLUTE VALUE OF THE PRODUCT OF THE PIVOTS IS"
90 PRINT "SMALLER THAN A NUMBER EPS, TO BE ENTERED BY THE"
100 PRINT "OPERATOR, THEN THE PROGRAM WILL END.": PRINT
110 PRINT "   PRESS A KEY TO CONTINUE."
120 A$ = INKEY$ : IF A$ = "" THEN 120
130 PRINT : PRINT "   NOW ENTER THE DATA AS:" : PRINT : PRINT
140 PRINT "2000 DATA N, EPS, A(1,1), A(2,1),..., A(N,1),"
150 PRINT "A(1,2),..., A(N,2), A(1,3),..., A(N-1,N), A(N,N)"
160 PRINT
170 PRINT "USE EXTRA DATA STATEMENT LINES IF NECESSARY."
180 PRINT
190 PRINT "   NOTE THAT THE ELEMENTS OF A ARE ENTERED BY COLUMN."
200 PRINT
210 PRINT "   THEN ENTER 'GOTO 240', AND HIT 'RETURN'." : PRINT
220 PRINT "USE MORE DATA STATEMENTS IF NECESSARY." : PRINT
230 END
240 DEFDBL A-K, O-Z : DEFINT I-N
250 DIM A(6,12), B(6,6), W(6), X(6), L(6)
260     REM L IS A VECTOR THAT KEEPS TRACK OF THE
270     REM LOCATIONS OF THE PIVOTS.
280 READ N, EPS
290 D = 1 : REM D WILL EVENTUALLY BECOME THE DETERMINANT OF A.
300 FOR J = 1 TO N
310    FOR I = 1 TO N
320       READ A(I,J)
330       A(I,J + N) = 0#
340    NEXT I
350    A(J,J + N) = 1#
360 NEXT J
370     REM IN LINES 380 TO 440 THE WEIGHTS, W(I), ARE FOUND.
380 FOR I = 1 TO N
390    L(I) = I
400    W(I) = ABS(A(I,1))
410    FOR J = 2 TO N
415       IF W(I) > ABS(A(I,J)) THEN 430
420       W(I) = ABS(A(I,J))
430    NEXT J
440 NEXT I
450     REM NOW START THE LARGE LOOP FOR SUCCESSIVE
460     REM ELIMINATIONS.
470 FOR K = 1 TO N - 1
```

```
480      REM SELECT AND IDENTIFY THE PIVOT.
490      REM J DEFINES THE PIVOT ROW.
500    RM = 0#
510    FOR I = K TO N
520       R = ABS(A(L(I),K))/W(L(I))
530       IF R < RM THEN 560
540       J = I
550       RM = R
560    NEXT I
570    LT = L(J) : L(J) = L(K) : L(K) = LT
580    D = D*A(LT,K) : IF ABS(D) < EPS THEN 980
590 REM PERFORM THE ELIMINATION
600    FOR I = K + 1 TO N
610       PF = A(L(I),K)/A(LT,K)
620       FOR J = K + 1 TO N + N
630          A(L(I),J) = A(L(I),J) - PF*A(LT,J)
640       NEXT J
650    NEXT I
660 NEXT K
670 D = D*A(L(N),N)
680 PRINT "THE ABSOLUTE VALUE OF THE DETERMINANT"
690 PRINT "IS " ; ABS(D)
700 IF ABS(D) < EPS THEN 980
710      REM BACK SUBSTITUTION
720 FOR J = N + 1 TO N + N
730    A(L(N),J) = A(L(N),J)/A(L(N),N)
740 NEXT J
750 FOR I = 1 TO N - 1
760    FOR J = N + 1 TO N + N
770       XT = A(L(N - I),J)
780       FOR K = N - I + 1 TO N
790          XT = XT - A(L(N - I),K)*A(L(K),J)
800       NEXT K
810       A(L(N - I),J) = XT/A(L(N - I),N - I)
820    NEXT J
830 NEXT I
840      REM REARRANGE AND PUT THE INVERSE INTO B.
850 FOR J = 1 TO N
860    FOR I = 1 TO N
870       B(I,J) = A(L(I),J + N)
880    NEXT I
890 NEXT J
900 PRINT : PRINT "COMPONENTS OF THE INVERSE MATRIX:"
910 PRINT
920 FOR I = 1 TO N
930    FOR J = 1 TO N
940       PRINT "B(" ; I ; "," ; J ; ") = " ; B(I,J)
950    NEXT J
960 NEXT I
970 END
980 PRINT "THE SYSTEM IS ILL-CONDITIONED, AND EXECUTION"
990 PRINT "HAS ENDED."
1000 END
2000 DATA 3 ,.0001,1,1,2,1,-1,1,1,1,-1,3,1,3
```

Appendix F

The Generation on the Computer of Gaussian Deviates

Functions that generate sample values of a *rectangular* probability distribution (usually, one where all numbers between 0 and 1 are equally probable) are routinely available in computer software. In order to model errors of observation, it is necessary to generate sample values of a Gaussian (or normal) distribution; these sample values are called Gaussian deviates. Gauss was able to deduce the distribution theoretically for errors of observation under the hypothesis that an error is formed by the simple addition of a large number of small independent elementary errors. In the program that follows, just twelve sample values of a rectangular distribution are added. The output, RV, is a sample value of the Gaussian distribution with r.m.s. value S, and mean value (usually assumed to be zero for observational errors) M. The syntax is IBM BASIC.

```
10 RANDOMIZE
20 RV = 0
30 FOR I = 1 TO 12
40 RV = RND + RV
50 NEXT I
60 RV = (RV - 6)*S + M
```

Appendix G

Some Orbits of Comets and Minor Planets

The following elements of cometary orbits are taken, with permission, from the *Catalogue of Cometary Orbits* prepared by B. G. Marsden (Ref. 71). The selection covers comets appearing over an interval of about two years.

	T ω	q Ω	e i
1980 I P/Honda-Mrkos-Pajdusakova (Seki-Halliday)	80 Apr. 11.0730 184.6326	0.580608 232.9286	0.808623 13.1184
1980 II Torres	80 Apr. 19.8741 334.9775	2.583929 278.8228	1.0 73.1449
1980 III P/Russell 2	80 May 19.5368 245.4411	2.160579 44.4534	0.416249 12.5313
1980 IV Cernis-Petrauskas	80 June 22.4409 337.8154	0.523249 159.9282	1.0 49.0739
1980 V P/Lovas	80 Sept. 3.4400 72.5699	1.675682 342.3256	0.614520 12.2928
1980 VI P/Forbes (Schuster)	80 Sept. 25.3200 262.5588	1.478871 23.0013	0.564961 4.6656
1980 VII P/Wild 3	80 Oct. 5.1804 179.3317	2.287518 72.0479	0.368027 15.4612
1980 VIII P/Reinmuth 1 (Schwartz-Shao)	80 Oct. 29.7420 9.4624	1.981564 121.1063	0.487103 8.3060
1980 IX P/Brooks 2 (Schuster)	80 Nov. 25.3926 198.2213	1.849711 176.2361	0.489786 5.5464
1980 X P/Stephan-Oterma (Schuster)	80 Dec. 5.1675 358.1610	1.574360 78.5122	0.860000 17.9809
1980 XI P/Encke	80 Dec. 6.5768 185.9827	0.339939 334.1956	0.846763 11.9460
1980 XII Meier	80 Dec. 9.6500 87.9642	1.519550 24.7375	0.994657 100.9806

	T ω	q Ω	e i
1980 XIII P/Tuttle (Shao-Schwartz)	80 Dec. 14.7054 206.8924	1.014937 269.8811	0.822575 54.4622
1980 XIV P/Harrington (Jekabsons)	80 Dec.24.0188 233.0384	1.604534 118.9527	0.555719 8.6516
1980 XV Bradfield	80 Dec. 29.5417 358.2855	0.259823 114.6465	0.999725 138.5882
1980u Panther	81 Jan. 27.3230 105.6034	1.657271 331.2992	0.998974 82.6419
1980n P/Reinmuth 2 (Jekabsons)	81 Jan. 29.9495 45.4068	1.945857 296.0492	0.454910 6.9692
1980i P/Borrelly (Schuster)	81 Feb. 20.0296 352.7716	1.319246 75.0577	0.631488 30.2008
1980l Russell	81 Mar. 6.3428 297.2715	2.110704 232.0736	1.0 128.7049
1979k P/Schwassmann- Wachmann 2 (Schwartz)	81 Mar. 17.0384 357.4594	2.135006 125.9424	0.386707 3.7298
1981g Gonzalez	81 Mar. 25.6616 181.6112	2.333601 143.2700	0.999395 107.1522
1980 r P/West-Kohoutek- Ikemura (Schuster)	81 Apr. 12.4548 358.1260	1.400759 84.6087	0.581428 30.0662
1980j P/Kohoutek (Schuster)	81 Apr. 17.6691 169.9183	1.570647 273.1213	0.536631 5.4163
1981k P/Howell	81 May 4.3582 214.6089	1.615399 75.3536	0.507452 5.5685
1981b P/Bus	81 June 11.3564 24.6446	2.182572 181.5278	0.374744 2.5783
1981e P/Finlay (Jekabsons-Candy)	81 June 20.0008 322.1293	1.100927 41.8014	0.698421 3.6425
1981d Bus	81 July 30.7016 189.8163	2.458484 23.5536	1.0 160.6597
1981c Elias	81 Aug. 18.2130 310.2393	4.742492 176.0068	1.000596 115.3169
1981a P/Longmore (Seki)	81 Oct. 21.5652 195.9250	2.399919 15.0088	0.342922 24.4131
1981f P/Gehrels 2 (Cochran)	81 Nov. 18.6674 183.4639	2.361641 215.5340	0.408480 6.6627
1981i P/Slaughter- Burnham (Schwartz-Shao)	81 Nov. 18.9049 44.1667	2.544226 345.9342	0.503950 8.1531

	T ω	q Ω	e i
1981j P/Swift-Gehrels (Shao-Schwartz)	81 Nov. 27.2669 84.5408	1.361128 314.0339	0.691291 9.2422
1981h P/Kearns/Kwee (Sheffer)	81 Nov. 30.5416 131.3808	2.223542 315.2621	0.485531 8.9855
1980b Bowell	82 Mar. 12.3120 134.8791	3.363815 114.0676	1.057196 1.6651
1982c P/du Toit- Hartley	82 Mar. 30.4450 251.6724	1.194664 308.5845	0.601961 2.9385
1982a P/Grigg- Skjellerup	82 May 14.9927 359.3282	0.989245 212.6324	0.665681 21.1366
1981l P/Vaisala 1 (Gibson)	82 July 30.6291 47.9336	1.799860 134.5116	0.633417 11.6069
1982e P/d'Arrest (Gibson-Everhart)	82 Sept. 14.3114 176.9682	1.291086 138.8598	0.624811 19.4301
1982 P/Gunn	82 Nov. 26.8618 196.9829	2.459149 67.8849	0.316361 10.3822
1982d P/Tempel 2 (Gibson)	83 June 1.5355 190.9220	1.381404 119.1579	0.544893 12.4375

Of those listed, the periodic comets Reinmuth 2, 1980n, and Gunn 1982 are particularly notable as having orbits that are, at times, strongly perturbed by Jupiter. You might find it of interest to investigate their orbits by numerical integration over several hundred years (forward and back in time). Some other comets with strongly perturbed orbits are:

	T ω	q Ω	e i
1958 IV P/Oterma	58 June 10.5008 354.8724	3.387823 155.1093	0.144488 3.9922
1965 VII P/de Vico- Swift	65 Aug. 23.2445 325.3516	1.624298 24.4216	0.524494 3.6064
1977 VII P/Gehrels 3	77 Apr. 32.2189 231.4802	3.424339 242.5502	0.151873 1.1013
1978 VIII P/Whipple	78 Mar. 27.5360 189.9764	2.468633 188.3387	0.352245 10.2461

Finally, here are some elements of Halley's comet for its most recent appearance. These were supplied by L. E. Doggett in April, 1986.

	T ω	q Ω	e i
1982i P/Halley	86 Feb. 9.45895 111.84618	0.5871122 58.14404	0.9672788 162.23957

For further information see "Long-term Evolution of Short-period Comets," Ref. 75.

The following minor planet elements are taken, with permission, from the *Astronomical Almanacs* of 1985 and 1986. Angles are in degrees.

(Equinox J2000.0)	i	Ω	ω
	a	e	M_0^*
1 Ceres	10.605	80.709	72.584
	2.7672	0.0784	28.881
2 Pallas	34.795	173.346	309.909
	2.7720	0.2337	334.594
3 Juno	12.998	170.577	246.897
	2.6677	0.2580	203.843
4 Vesta	7.138	104.069	150.792
	2.3625	0.0897	76.694
433 Eros	10.832	304.496	178.510
	1.4582	0.2229	170.451
1566 Icarus	22.896	88.195	31.182
	1.0779	0.8268	72.091
3415 Danby**	1.355	253.838	137.856
	3.9680	0.2489	101.805

*Mean anomaly at 1986 June 19.0, with the exception of Pallas, for which the epoch is 1985 December 1.0.

**Minor planet 3415 was discovered in 1928 by K. Reinmuth. It was named after the author in 1986. The name also is for his daughter, Dinah, who was an assiduous worker at the Smithsonian Astrophysical Observatory while she was an undergraduate at Harvard.

Appendix H

The Greek Alphabet

	Sign			Sign	
Name	Capital	Small	Name	Capital	Small
Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π or ϖ
Epsilon	E	ϵ or ε	Rho	P	ρ or ϱ
Zeta	Z	ζ	Sigma	Σ	σ or ς
Eta	H	η	Tau	T	τ
Theta	Θ	θ or ϑ	Upsilon	Υ	υ
Iota	I	ι	Phi	Φ	ϕ or φ
Kappa	K	κ	Chi	X	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	M	μ	Omega	Ω	ω

Appendix I

Random Variables, and Least Squares

Let ζ represent *a random variable* or *variate*. This might be the error of a particular observation (range-rate, for instance) made by a particular instrument.

Let z represent a *sample value* of ζ.

The *sample space* of ζ is the set of all possible sample values; here, we shall take it to be $-\infty < z < \infty$.

The *probability distribution function* of ζ is

$$F_\zeta(z) = Pr[\{z' ; z' \leq z\}]$$

i.e., "the probability that a sample point z' has a value less than or equal to a specified number z." In particular, $F(-\infty) = 0$, and $F(\infty) = 1$. (A probablility of 1 stands for certainty.)

Assuming $F_\zeta(z)$ to be differentiable, the *probablility density function* of ζ is

$$\phi_\zeta(z) = \frac{dF_\zeta(z)}{dz}.$$

From the fundamental theorem of calculus we have

$$F_\zeta(z) = \int_{-\infty}^{z} \phi_\zeta(z')dz'.$$

Then also

$$\phi_\zeta(z)dz = Pr[\{z' ; z < z' < z + dz\}],$$

i.e., the probablity that a sample point z' lies in the interval $z < z' < z + dz$. Also

$$\int_b^a \phi_\zeta(z')dz' = Pr[(z' ; b < z' < a)].$$

In particular,

$$\int_{-\infty}^{\infty} \phi_\zeta(z')dz' = 1. \tag{I.1}$$

The *expected value* (or *first moment* or *statistical average*) of ζ is

$$E[\zeta] = \int_{-\infty}^{\infty} z\phi_\zeta(z)dz. \tag{I.2}$$

This is the *bias* of ζ, often referred to as *systematic error*. The variate is said to be *unbiased* if $E[\phi] = 0$.

Knowledge of the "spread" of the probability density is given by the *second moment* or *variance*:

$$\sigma^2 = E[(\zeta - E[\zeta])^2] = \int_{-\infty}^{\infty} (z - E[\zeta])^2 \phi_\zeta(z)dz. \tag{I.3}$$

σ is called the *root mean square deviation*.

The phrase "the expected value of...," or its corresponding symbolic representation, $E[\ldots]$, can be considered as an operator. If $f(\zeta)$ is a non-random function, then its expected value is defined to be

$$E[f(\zeta)] = \int_{-\infty}^{\infty} f(z)\phi_\zeta(z)dz. \tag{I.4}$$

A function of a random variable is itself a random variable. In particular, since observations are random variables, so are the parameters that are estimated through the use of the observations.

The *normal distribution* is defined by

$$\phi_\zeta(z) = \frac{1}{\sqrt{2\pi}h} \exp\left(-\frac{1}{2}\frac{(z - z_0)^2}{h^2}\right). \tag{I.5}$$

Then $E[\zeta] = z_0$, and $\sigma^2 = h^2$. Of the two parameters h and z_0, h is taken to be strictly positive. The factor preceding the exponential is calculated so that (I.1) should be true. For large h the distribution has a relatively low maximum, and a wide spread; for small h the distribution falls off sharply form the maximum.

The normal distribution is often called the *Gaussian distribution*, and the corresponding variate ζ is a *Gaussian variate*. It dates back to its invention by de Moivre, in 1733, but acquired its present importance in the hands of Gauss in 1809. Gauss deduced the distribution theoretically for errors under the hypothesis that an error is formed by the simple addition of small independent elementary errors; these elementary errors are of the same order of magnitude and are as likely to be positive as negative.

Next, consider two random variables η and ζ, with sample values y and z, respectively. The *joint probability function* is

$$F_{\eta\zeta}(y, z) = Pr[\{y', z'; y' \le y, z' \le z\}].$$

i.e., the probablility that sample values y' and z' satisfy both $y' \le y$ and $z' \le z$ for specified y and z. The *joint density function* is

$$\phi_{\eta\zeta}(y, z) = \frac{\partial^2 F_{\eta\zeta}(y, z)}{\partial y\, \partial z},$$

assuming that all necessary derivatives exist.

The first moments are

$$\left.\begin{aligned} \mu_\eta &= E[\eta] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\phi_{\eta\zeta}(y, z)dy\, dz \\ &\text{and} \\ \mu_\zeta &= E[\zeta] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\phi_{\eta\zeta}(y, z)dydz. \end{aligned}\right] \tag{I.6}$$

Second moments are

$$\left.\begin{aligned} \sigma_\eta^2 &= E[(\eta - E[\eta])^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y - E[\eta])^2\phi_{\eta\zeta}(y, z)dy\, dz, \\ \sigma_\zeta^2 &= E[(\zeta - E[\zeta])^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(z - E[\zeta])^2\phi_{\eta\zeta}(y, z)dy\, dz, \\ &\text{and} \\ \mu_{\eta\zeta} &= E[(\eta - E[\eta])(\zeta - E[\zeta])] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y - E[\eta])(z - E[\zeta])\phi_{\eta\zeta}(y, z)dy\, dz. \end{aligned}\right] \tag{I.7}$$

If the variates are independent, so that we can write

$$\phi_{\eta\zeta}(y, z) = \phi_\eta(y)\, \phi_\zeta(z),$$

then the variates are called *statistically independent*. In this case, $\mu_{\eta\zeta} = 0$.

Correlation between η and ζ is specified by the number

$$\rho = \frac{\mu_{\eta\zeta}}{\sigma_\eta\, \sigma_\zeta} \tag{I.8}$$

called the *linear correlation coefficient*. If the variates are statistically independent, $\rho = 0$; if they are linearly related, then $\rho = \pm 1$.

The normal distribution can be generalized for a bivariate distribution in terms of its first and second moments as follows.

Let

$$Q = \begin{bmatrix} \sigma_\eta^2 & \mu_{\eta\zeta} \\ \mu_{\eta\zeta} & \sigma_\zeta^2 \end{bmatrix}. \tag{I.9}$$

This is called the *covariance matrix* of the distribution. Then we can write

$$\phi_{\eta\zeta}(y, z) = \frac{1}{2\pi\, |Q|^{1/2}} \exp\left(-\frac{1}{2}[y - \mu_\eta \;\; z - \mu_\zeta]Q^{-1}\begin{bmatrix} y - \mu_\eta \\ z - \mu_\zeta \end{bmatrix}\right). \tag{I.10}$$

$|Q|$ is the determinant of Q.

This form is convenient for the generalization to any number of joint variates. Let $\boldsymbol{\zeta}$ denote a column matrix, or vector, having components $\zeta_1, \zeta_2, \ldots, \zeta_n$. The probablility density function is a scalar function $\phi_{\boldsymbol{\zeta}}(\mathbf{z})$, $\mathbf{z}$ now being a vector. The expected value of $\mathbf{z}$ is a vector having as its i-th component

$$E[\zeta_i] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} z_i \phi_\zeta(\mathbf{z})\, dz_1\, dz_2 \ldots dz_n,$$

so the whole vector can be conveniently written as

$$E[\boldsymbol{\zeta}] = \int_{-\infty}^{\infty} \mathbf{z}\phi_{\boldsymbol{\zeta}}(\mathbf{z})\, d\mathbf{z}. \tag{I.11}$$

A typical second order moment is

$$\mu_{ij} = E[(\zeta_i - E[\zeta_i])(\zeta_j - E[\zeta_j])],$$

so that all the second moments are contained in the symmetric matrix

$$Q = E[\boldsymbol{\zeta} - E[(\boldsymbol{\zeta}])(\boldsymbol{\zeta} - E[\boldsymbol{\zeta}])^T], \tag{I.12}$$

which is the *covariance matrix* of the distribution. It has elements μ_{ij}, where $\mu_{ii} = \sigma_i^2$.

The *correlation matrix* has typical element

$$\rho_{ij} = \frac{\mu_{ij}}{\sigma_i \sigma_j}. \tag{I.13}$$

The *normal distribution* in this case is

$$\phi_{\boldsymbol{\zeta}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}|Q|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{z} - E[\boldsymbol{\zeta}])^T Q^{-1}(\mathbf{z} - E[\boldsymbol{\zeta}])\right). \tag{I.14}$$

Now consider the problem of estimating the m components of $\mathbf{X}$, the parameters specifying an orbit, from the n components of $\mathbf{Y}$, the observations. Given $\mathbf{X}$, observations free of error can be found from

$$\mathbf{Y}_c = \mathbf{Y}(\mathbf{X}). \tag{I.15}$$

Let $\boldsymbol{\zeta}$ be the variate denoting the observational error, having a normal distribution without bias:

$$\phi_{\boldsymbol{\zeta}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}|Q|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{z}^T Q^{-1}\mathbf{z}\right). \tag{I.16}$$

Making an estimate of $\mathbf{X}$ is equivalent to estimating the error of $\mathbf{Y}$. The *likelihood* of a given estimate $\mathbf{X}$ is

$$L(\mathbf{X}) = \frac{1}{(2\pi)^{n/2}|Q|^{1/2}} \exp\left(-\frac{1}{2}[(\mathbf{Y}_o - \mathbf{Y}_c(\mathbf{X})]^T Q^{-1}[\mathbf{Y}_o - \mathbf{Y}_c(\mathbf{X})]\right). \tag{I.17}$$

where $\mathbf{Y}_o$ contains the actual observations. The "best" estimate of $\mathbf{X}$ is that which maximizes the likelihood. In the present case, this is equavelent to minimizing the *variance*, or the argument of the exponential. Let the vector operator $\nabla_{\mathbf{X}}$ be defined by

$$\nabla_{\mathbf{X}} = \begin{bmatrix} \partial/\partial X_1 \\ \partial/\partial X_2 \\ \vdots \\ \partial/\partial X_m \end{bmatrix}.$$

Then for $L(\mathbf{X})$ to have a maximum at $\mathbf{X} = \mathbf{X}^*$, it is necessary that

$$\left(\nabla_{\mathbf{X}}\Big[\mathbf{Y}(\mathbf{X})\Big]^T Q^{-1}\Big[\mathbf{Y}_o - \mathbf{Y}_c(\mathbf{X})\Big]\right)_{\mathbf{X}=\mathbf{X}^*} = 0. \tag{I.18}$$

This equation is solved through leariation, as in section (7.5). We let

$$\mathbf{y} = \mathbf{Y}_o - \mathbf{Y}_c(\mathbf{X}). \tag{I.19}$$

Also

$$\begin{aligned} \nabla_{\mathbf{X}}[\mathbf{Y}_c(\mathbf{X})]^T &= \begin{bmatrix} \partial/\partial X_1 \\ \partial/\partial X_2 \\ \vdots \\ \partial/\partial X_m \end{bmatrix} [Y_1(\mathbf{X})\; Y_2(\mathbf{X}) \ldots Y_n(\mathbf{X})] \\ &= \begin{bmatrix} \partial Y_1/\partial X_1 & \partial_2/\partial X_1 & \cdots & \partial Y_n/\partial X_1 \\ \partial Y_1/\partial X_2 & \partial_2/\partial X_2 & \cdots & \partial Y_n/\partial X_2 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \partial Y_1/\partial X_m & \partial_2/\partial X_m & \cdots & \partial Y_n/\partial X_m \end{bmatrix} \\ &= J^T, \end{aligned} \tag{I.20}$$

where J is the Jacobian matrix $J = [\partial\mathbf{Y}/\partial\mathbf{X}]$. Also

$$\begin{aligned} \mathbf{Y}_o - \mathbf{Y}_c(\mathbf{X}^*) &= \mathbf{Y}_o - \mathbf{Y}_c(\mathbf{X} + \mathbf{x}^*) \\ &= \mathbf{Y}_o - \{\mathbf{Y}_c(\mathbf{X}) + J\mathbf{x}^* + \cdots\} \\ &\approx \mathbf{y} - J\mathbf{x}^*, \end{aligned} \tag{I.21}$$

where terms of order $(\mathbf{x}^*)^2$ are neglected. Then the linearized form of (I.18) becomes

$$J^T Q^{-1}(\mathbf{y} - J\mathbf{x}^*) = \mathbf{0}$$

or

$$J^T Q^{-1} J\mathbf{x}^* = J^T Q^{-1}\mathbf{y}. \tag{I.22}$$

These are the *normal equations*

The best estimate x^* is found from the solution of (F.22), i.e.,

$$\mathbf{x}^* = (J^T Q^{-1} J)^{-1} J^T Q^{-1} y = B\mathbf{y}, \tag{I.23}$$

where B is a *linear estimator.* $\mathbf{x}$ is a sample value of a random variable $\boldsymbol{\xi}$. To judge the usefulness of an estimate, we need its statistics, and in particular we need the covaniance matrix of $\boldsymbol{\xi}$:

$$\begin{aligned} P &= E\left[\boldsymbol{\xi}\boldsymbol{\xi}^T\right] \\ &= E\left[(B\zeta)(B\zeta)^T\right] \\ &= BE\left[\zeta\zeta^T\right]B^T \\ &= BQB^T \\ &= (J^T Q^{-1} J)^{-1}. \end{aligned} \tag{I.24}$$

Appendix J

Notes on Hamiltonian Mechanics

For further details see Refs. 22, 26, and 27 for excellent introductory material and Ref. 19 for applications.

J.1 Elements of Lagrangian Mechanics

In this introductory summary we consider mechanical systems in which the forces can be derived from a potential. A system can be specified by a set of "generalized coordinates." For instance, a rigid body requires six coordinates, three for the location of its center of mass and three for its orientation. These may be written as conventional coordinates, or specified by the set $q_1, q_2 \cdots, q_n$ of *generalized coordinates*. The system is then said to have n degrees of freedom. These are assumed to be independent; we ignore the possibility of constraints, i.e., relations between the coordinates.

We can then write down an expression for the kinetic energy, T, in terms of these coordinates and their derivatives. For a single particle of mass m, in three dimensions,

$$x = q_1,\ y = q_2,\ z = q_3; \quad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

Or, using the spherical polar coordinates of section (5.6),

$$r = q_1, \theta = q_2, \phi = q_3,$$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\dot{\phi}^2). \tag{J.1.1}$$

We assume that the dynamical system can be analyzed in terms of a system of masses, using, at first, Cartesian coordinates. Let each of these coordinates be expressed in terms of the generalized coordinates $q_1, q_2, \cdots, q_n$, and possibly the time t. Then

$$\frac{dx}{dt} = \dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial x}{\partial q_n}\dot{q}_n,$$

so that

$$\frac{\partial \dot{x}}{\partial \dot{q}_k} = \frac{\partial x}{\partial q_k}.$$

Therefore,

$$\frac{d}{dt}\left(\frac{1}{2}m\frac{\partial \dot{x}^2}{\partial \dot{q}_k}\right) = m\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_k}\right) = X\frac{\partial x}{\partial q_k} + m\dot{x}\frac{\partial \dot{x}}{\partial q_k}, \qquad \text{(J.1.2)}$$

where X is the x-component of force acting on m. Now

$$T = \frac{1}{2}\sum m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right).$$

So, adding all equations like (J.1.2) for all particles and all coordinates,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum\left(X\frac{\partial x}{\partial q_k} + Y\frac{\partial y}{\partial q_k} + Z\frac{\partial z}{\partial q_k}\right) + \frac{\partial T}{\partial q_k}. \qquad \text{(J.1.3)}$$

Suppose that the forces arise from a potential V. Then

$$\sum(X dx + Y dy + Z dz) = -dV.$$

Then the right hand side of (J.1.3) can be written as

$$\frac{\partial (T - V)}{\partial q_k}.$$

Now define the "Lagrangian", or "Lagrangian function"

$$L = T - V \qquad \text{(J.1.4)}$$

We then have the set of Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, 2, \cdots, n. \qquad \text{(J.1.5)}$$

If L does not contain the time explicitly, then we have the integral

$$\sum \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = h. \qquad \text{(J.1.6)}$$

This can also be written as $T + V = h$; so h is the total energy.

J.2 Hamilton's Equations

Define the "monenta"

$$p_k = \frac{\partial L}{\partial \dot{q}_k}. \qquad \text{(J.2.1)}$$

From Lagrange's equations,

$$\dot{p}_k = \frac{\partial L}{\partial q_k}. \qquad \text{(J.2.2)}$$

Provided that the Hessian $|\frac{\partial^2 L}{\partial q_i \partial q_j}| \neq 0$, the equations (J.2.1) can be solved for the $\dot{q}_k$ in terms of the coordinates and monenta. Now define the "Hamiltonian"

$$H(q_1, \cdots, q_n, p_1, \cdots p_n, t) = \sum \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L, \qquad \text{(J.2.3)}$$

where we assume that the substitution for the $\dot{q}_k$ has been made.

Consider L geometrically in terms of the coordinates and momenta, we can write down the two differential forms:

$$dL = \sum \frac{\partial L}{\partial q_k} dq_k + \sum \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k,$$

and

$$d\left(\sum \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) = \sum \dot{q}_k dp_k + \sum \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k.$$

Then, using (J.2.2), we find

$$dH = \sum(\dot{q}_k dp_k - \dot{p}_k dq_k). \qquad \text{(J.2.4)}$$

From this, it follows that

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \ \dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad k = 1, 2, \cdots, n. \qquad \text{(J.2.5)}$$

These are Hamilton's equations of motion, often called "canonical equations."

If H does not contain the time explicitly, then

$$H = h$$

a constant: H is the total energy.

If x is the column vector $x = [q_1 \ q_2 \cdots q_n \ p_1 \ p_2 \cdots p_n]^T$,

$$J = \begin{bmatrix} O_n & I_n \\ -I_n & O_n \end{bmatrix}, \qquad \text{(J.2.6)}$$

and

$$H_x = \begin{bmatrix} \partial H/\partial x_1 \\ \partial H/\partial x_2 \\ \vdots \\ \partial H/\partial x_{2n} \end{bmatrix},$$

then the canonical equations can be written as

$$\dot{x} = J H_x. \qquad \text{(J.2.7)}$$

Suppose that we have a time-dependent Hamiltonian

$$H(q_1,\ \cdots,\ q_n,\ p_1,\ \cdots,\ p_n,\ t).$$

Let $q_{n+1} = t$, and define

$$K(q_1,\ \cdots,\ q_n,\ q_{n+1},\ p_1,\ \cdots,\ p_n,\ p_{n+1}) = H + p_{n+1}. \tag{J.2.8}$$

The canonical equations for the Hamiltonian K contain those for the Hamiltonian H. This means that results valid for time-independent (conservative) Hamiltonians can be easily generalized to time-dependent Hamiltonians.

J.3 Canonical Transformations

Suppose that a transformation from x to y is given by

$$y_k = y_k(x_1, x_2, \cdots, x_{2n}),\quad k = 1,\ 2,\ \cdots,\ 2n. \tag{J.3.1}$$

Here we shall assume that all functions have continuous second derivatives in the region of interest. For this transformation we have the Jacobian matrix:

$$M = \frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_{2n}} \\ \vdots & \vdots\vdots\vdots & \vdots \\ \frac{\partial y_{2n}}{\partial x_1} & \cdots & \frac{\partial y_{2n}}{\partial x_{2n}} \end{bmatrix} \tag{J.3.2}$$

If at some point the Jacobian $|M| \neq 0$, then in the neighborhood of that point equations (J.3.1) can be inverted uniquely, giving

$$x_k = x_k(y_1,\ y_2, \cdots, y_{2n}),\quad k = 1,\ 2,\ \cdots,\ 2n. \tag{J.3.3}$$

The Jacobian matrix of this transformation is M^{-1}.

Consider a transformation from $(q,\ p)$ to $(Q,\ P)$, with Jacobian non-zero at all points in a given region, so that the transformation is invertible in that region. This transformation is said to be canonical if for *every* Hamiltonian function $H(q, p, t)$ there exists another function $K(Q, P, t)$ such that the transformation carries the canonical equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k},\ \dot{p}_k = -\frac{\partial H}{\partial q_k},\ k = 1,\ 2,\ \cdots,\ n \tag{J.3.4}$$

into

$$\dot{Q}_k = \frac{\partial K}{\partial P_k},\ \dot{P}_k = -\frac{\partial K}{\partial Q_k},\ k = 1,\ 2,\ \cdots,\ n. \tag{J.3.5}$$

Wintner (Ref. 44) pp. 22-26, proves the following theorem: This transformation is canonical if and only if

$$M^T J M = \mu J \tag{J.3.6}$$

for $\mu \neq 0$. M is the Jacobian matrix of the transformation and J is defined in (J.2.6). μ is called the "multiplier" of the transformation; the possibility that $\mu \neq 1$ is ignored in most textbooks.

If (J.3.6) is satisfied with $\mu = 1$, then M is said to be ***symplectic***. We shall show that $M^T J M = J$ is a sufficient condition for the transformation to be canonical. Let

$$x = \begin{bmatrix} q \\ p \end{bmatrix} \text{ and } y = \begin{bmatrix} Q \\ P \end{bmatrix}$$

Then

$$\dot{y} = M\dot{x} \text{ and } \frac{\partial H}{\partial x} = M^T \frac{\partial K}{\partial y},\ \text{ where } K(y) = Hx(y),$$

or the new Hamiltonian is the same as the old. (We continue to work with time-independent Hamiltonians.) Then, since $\dot{x} = JH_x$,

$$\dot{y} = MJH_x = MJM^T K_y = JK_y$$

since J is symplectic. This is what we had to show.

A condition for a canonical transformation is sometimes given in terms of *Lagrange brackets* or *Poisson brackets*. First, we comment that if M is symplectic, then so is $M^{-1} = \partial x/\partial y$. Then

$$J = \left(\frac{\partial x}{\partial y}\right)^T J \left(\frac{\partial x}{\partial y}\right) = \begin{bmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{bmatrix}^T \begin{bmatrix} O_n & I_n \\ -I_n & O_n \end{bmatrix} \begin{bmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{m=1}^{n}\left(\frac{\partial q_m}{\partial Q_j}\frac{\partial p_m}{\partial Q_k} - \frac{\partial p_m}{\partial Q_j}\frac{\partial q_m}{\partial Q_k}\right) & \sum_{m=1}^{n}\left(\frac{\partial q_m}{\partial Q_j}\frac{\partial p_m}{\partial P_k} - \frac{\partial p_m}{\partial Q_j}\frac{\partial q_m}{\partial P_k}\right) \\ \sum_{m=1}^{n}\left(\frac{\partial q_m}{\partial P_j}\frac{\partial p_m}{\partial Q_k} - \frac{\partial p}{\partial P_j}\frac{\partial q_m}{\partial Q_k}\right) & \sum_{m=1}^{n}\left(\frac{\partial q_m}{\partial P_j}\frac{\partial p_m}{\partial P_k} - \frac{\partial p_m}{\partial P_j}\frac{\partial q_m}{\partial P_k}\right) \end{bmatrix}$$

$$= \begin{bmatrix} [Q_j, Q_k] & [Q_j, P_k] \\ [P_j, Q_k] & [P_j, P_k] \end{bmatrix}.$$

This is the Lagrange matrix, and its components are Lagrange brackets. Its inverse is the Poisson matrix. A sufficient condition for a canonical transformation is, therefore,

$$[Q_j, Q_k] = [P_j, P_k] = O_n,\ [Q_j, P_k] = I_n,\ [P_j, Q_k] = -I_n,\ j, k, = 1, 2, \cdots, n. \tag{J.3.7}$$

Next, consider the differential form

$$\sum_{k=1}^{n}(p_k dq_k - P_k dQ_k), \tag{J.3.8}$$

and express it in terms of Q_k and P_k. It becomes

$$\sum_{k=1}^{n}\left(p_k(Q,P)\left(\sum_{j=1}^{n}\left(\frac{\partial q_k}{\partial Q_j}dQ_j + \frac{\partial q_k}{\partial P_j}dP_j\right)\right) - P_k dQ_k\right)$$

$$= \sum_{j=1}^{n}\left(\left(\sum_{k=1}^{n}P_k\frac{\partial q_k}{\partial P_j}\right)dP_j + \left(\sum_{k=1}^{n}P_k\frac{\partial q_k}{\partial Q_j} - P_j\right)dQ_j\right) = K.$$

It is a long piece of work, but it is possible to show that the conditions that K be an exact differential form are all met using the properties of the Lagrange brackets given in (J.3.7). Consequently the transformation is canonical if and only if there is a function $W(Q,P)$ whose total drivative is K.

J.4 Canonical Transformations Defined by Functions

Let us start with a function $S(q,Q)$ where the arguments are the old and new coordinates in a mapping from $(q,\ p)$ to $(Q,\ P)$. S is assumed to have continuous second derivatives. Then

$$dS = \sum_{k=1}^{n}\left(\frac{\partial S}{\partial q_k}dq_k + \frac{\partial S}{\partial Q_k}dQ_k\right). \tag{J.4.1}$$

Let the mapping be *defined* by the relations

$$p_k = \frac{\partial S}{\partial q_k},\ P_k = -\frac{\partial S}{\partial Q_k},\ k = 1,\ 2,\ \cdots,\ n. \tag{J.4.2}$$

This procedure guarantees that the differential form

$$K_1 = \sum_{k=1}^{n}(p_k dq_k - P_k dQ_k) \tag{J.4.3}$$

in exact, so that the transformation is canonical. (We also need a condition that the equations (J.4.2) can be solved for q and p in terms of Q and P. This is the case if

$$\left|\frac{\partial^2 S}{\partial q_j\,\partial Q_k}\right| \neq 0$$

in the region of interest.)

The choice of q and Q as arguments of the generating function, S, is not essential. If K_1 is exact, then so also will be

$$K_2 = \sum_{k=1}^{n}(q_k dp_k + P_k dQ_k) = d\left(\sum_{k=1}^{n}(p_k q_k)\right) - K_1.$$

This leads to the canonical transformation:

$$S = S(p,Q):\ \ q_k = \frac{\partial S}{\partial p_k},\ P_k = \frac{\partial S}{\partial Q_k},\ k = 1,\ 2,\ \cdots,\ n. \tag{J.4.4}$$

Similarly, we have two other exact forms K_3 and K_4.

$$K_3 = \sum_{k=1}^{n}(p_k dq_k + Q_k dP_k) = d\left(\sum_{k=1}^{n}(P_k Q_k)\right) + K_1 \tag{J.4.5}$$

leads to

$$S = S(q,p): p_k = \frac{\partial S}{\partial p_k},\ Q_k = -\frac{\partial S}{\partial P_k},\ k = 1,\ 2,\ \cdots,\ n. \tag{J.4.6}$$

and

$$K_4 = \sum_{k=1}^{n}(q_k dp_k - Q_k dP_k) = d\left(\sum_{k=1}^{n}(p_k q_k)\right) - K_3 \tag{J.4.7}$$

leads to

$$S = S(p,P):\ \ q_k = \frac{\partial S}{\partial p_k},\ \ Q_k = -\frac{\partial S}{\partial P_k},\ k = 1, 2, \cdots, n. \tag{J.4.8}$$

J.5 The Hamilton-Jacobi Equation

Consider the time-dependent Hamiltonian

$$H(q_1, \cdots, q_n,\ p_1, \cdots, p_n,\ t).$$

As before, let $q_{n+1} = t$, and introduce the new Hamiltonian

$$K(q_1, \cdots, q_n,\ q_{n+1},\ p_1, \cdots, p_n, p_{n+1}) = H + p_{n+1}.$$

Apply to this the transformation induced by the generating function

$$S(q_1, \cdots, q_n, q_{n+1},\ Q_1, \cdots, Q_n),$$

where Q_{n+1} does not appear. For the time-independent Hamiltonian K, we know that the new and old Hamiltonians will be equal. As part of the transformation,

$$P_{n+1} = \frac{\partial S}{\partial q_{n+1}} = \frac{\partial S}{\partial t},$$

so if we return to the original notation, we have the following summary:

Hamiltonian: $H(q,\ p,\ t)$.

Generating function: $S(q,\ Q,\ t)$, where $p = \frac{\partial S}{\partial q}$, $P = -\frac{\partial S}{\partial Q}$.

New Hamiltonian: $H^* = H + \frac{\partial S}{\partial t}$.

A motivation for carrying out a canonical transformation is that the new canonical equations will be easier to solve than the old. Carrying this to an extreme, we consider the possibility that $H^* = 0$. Then Q_k and P_k will all be constants, so we set

$$Q_k = \alpha_k,\ P_k = \beta_k,\ k = 1,\ 2,\ \cdots,\ n. \tag{J.5.1}$$

Now

$$p_k = \frac{\partial S}{\partial q_k},\ k = 1,\ 2,\ \cdots,\ n,$$

so S must satisfy the partial differential equation

$$H\left(q,\ \frac{\partial S}{\partial q},\ t\right) = 0. \tag{J.5.2}$$

This is the *Hamilton-Jacobi equation.* We are not, fortunately, looking for its general solution; we just want a solution

$$S(q_1,,\cdots,q_n,\alpha_1,\cdots,\alpha_n,t) \tag{J.5.3}$$

which contains n independent arbitrary constants, $\alpha_1,\cdots,\alpha_n$. Once found, S determines the transformation

$$P_k = \frac{\partial S}{\partial q_k},\ \beta_k = -\frac{\partial S}{\partial \alpha_k},\ k = 1,\ 2,\ \cdots,n. \tag{J.5.4}$$

These equations actually contain the solution of the problem.

J.6 The Problem of Two Bodies

Using spherical polar coordinates, and equations (J.1.1), (J.2.1) and (J.2.3), we have

$$q_1 = r,\ q_2 = \theta,\ q_3 = \phi,\ p_1 = \dot{r},\ p_2 = r^2\dot{\theta},\ p_3 = r^2\cos^2\theta\dot{\phi},$$

$$H = \frac{1}{2}\left(p_1^2 + \frac{p_2^2}{r^2} + \frac{p_3^2}{r^2\cos^2\theta}\right) - \frac{\mu}{r}. \tag{J.6.1}$$

The Hamilton-Jacobi equation is

$$\frac{1}{2}\left(S_r^2 + \frac{S_\theta^2}{r^2} + \frac{S_\phi^2}{r^2\cos^2\theta}\right) - \frac{\mu}{r} + S_t = 0, \tag{J.6.2}$$

where the subscripts denote partial differentiation. The equation is solved in the form

$$S = S_0(t) + S_1(r) + S_2(\theta) + S_3(\theta).$$

Details can be found in Refs 19, 26, 29. The solution can be written as

$$S = -\alpha_1 t + \alpha_3\phi + \int_{r_o}^{r}\left(2\alpha_1 + \frac{2\mu}{r} - \frac{a_2^2}{r^2}\right)^{-1/2} dr$$
$$+ \int_0^\theta (\alpha_3^2 - \alpha_3^2\sec^2\theta)^{1/2} d\theta. \tag{J.6.3}$$

We are not concerned with the details of the solution, describing the actual motion, but with the interpretation of the canonical constants:

$$\left.\begin{aligned} \alpha_1 &= -\mu/2a, & \beta_1 &= T,\\ \alpha_2 &= \sqrt{a\mu(1-e^2)}, & \beta_2 &= -\omega,\\ \alpha_3 &= \sqrt{a\mu(1-e^2)}\cos i, & \beta_3 &= -\Omega. \end{aligned}\right] \tag{J.6.4}$$

A disadvantage of this set is that since the mean motion is a function of a, some derivatives result in the appearance of the time outside trigonometric terms. This is avoided in another set introduced by Delaunay. First, consider the mapping from

$$(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\ \text{to}\ (A_1,A_2,A_3,B_1,B_2,B_3)$$

given by the generating function

$$S = \alpha_1 t - \mu(-2\alpha_1)^{-1/2}A_1 - \alpha_2 A_2 - \alpha_3 A_3. \tag{J.6.5}$$

Then

$$B_1 = -\frac{\partial S}{\partial A_1} = \sqrt{a\mu},\ \ B_2 = \alpha_2,\ \ B_3 = \alpha_3.$$

$$T = \beta_1 = \frac{\partial S}{\partial \alpha_1} = t - \mu(-2\alpha_1)^{-3/2}A_1 = t - \frac{1}{n}A_1,$$

so $A_1 = n(t-T)$, the mean anomaly. Also $-\omega = \beta_2 = -A_2$, and $-\Omega = \beta_3 = -A_3$. The new Hamiltonian is

$$\frac{\partial S}{\partial t} = \alpha_1 = -\frac{\mu^2}{2L^2},$$

where $L = B_1$.

Finally, according to astronomical practice and tradition, we switch coordinates and momenta — the equivalent of changing the sign of the Hamiltonian.

We shall call this new Hamiltonian F. Then in Delaunay's notation, we have the following canonical set:

$$\left.\begin{aligned} L &= \sqrt{a\mu}, & \ell &= M, \\ G &= \sqrt{a\mu(1-e^2)}, & g &= \omega, \\ H &= \sqrt{a\mu(1-e^2)}\cos i, & h &= \Omega, \end{aligned}\right] \tag{J.6.6}$$

$$F_0 = \frac{\mu^2}{2L^2}.$$

J.7 Perturbed Motion

Suppose that the original equations of motion are

$$\ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = \nabla F_1. \tag{J.7.1}$$

Let

$$F = F_0 + F_1. \tag{J.7.2}$$

The corresponding canonical equations will be

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial \ell} & \frac{d\ell}{dt} &= -\frac{\partial F}{\partial L} \\ \frac{dG}{dt} &= \frac{\partial F}{\partial g} & \frac{dg}{dt} &= -\frac{\partial F}{\partial G} \\ \frac{dH}{dt} &= \frac{\partial F}{\partial h} & \frac{dh}{dt} &= -\frac{\partial F}{\partial H} \end{aligned}\right] \tag{J.7.3}$$

Equations (11.9.9) can be derived from these.

For a simple application consider Keplerian motion perturbed by a force varying inversely with the cube of r; then we can take

$$F_1 = \frac{\varepsilon}{r^2}. \tag{J.7.4}$$

The motion will be planar, so only the variables (L, G, l, g,) need to be considered. Also, F_1 is a function of L, G, and l, but not of g. We shall make a transformation from (L, G, l, g)-to (L^*, G^*, l^*, g^*) using a generating function $S(L^*, G^*, l, g)$. This will be expanded in powers of ε:

$$S = S_0(L^*, G^*, l, g) + \varepsilon S_1(L^*, G^*, l, -) + \cdots \tag{J.7.5}$$

The purpose of the transformation is to make the new Hamiltonian, $F^*(L^*, G^*)$ independent of l^*. Thus L^* and G^* will be constants and l^* and g^* will vary secularly with the time. For $\varepsilon = 0$, the old and new coordinates would be the same, so

$$S_0 = L^* l + G^* g, \tag{J.7.6}$$

Also if F^* is expanded similarly in powers of ε, then

$$F_0^*(L^*) = F_0(L^*).$$

The full transformation is

$$\left.\begin{aligned} L &= \frac{\partial S}{\partial l} = L^* + \varepsilon\frac{\partial S_1}{\partial l} + \cdots, & G &= \frac{\partial S}{\partial g} = -G^*, \\ l^* &= \frac{\partial S}{\partial L^*} = l + \varepsilon\frac{\partial S_1}{\partial L^*} + \cdots, & g^* &= \frac{\partial S}{\partial G^*} = g + \varepsilon\frac{\partial S_1}{\partial G^*} + \cdots \end{aligned}\right] \tag{J.7.7}$$

Equating the new and old Hamiltonians,

$$\begin{aligned} &F_0^*(L^*) + \varepsilon F_1^*(L^*, G^*) + \cdots = \\ &\qquad F_0\left(L^* + \varepsilon\frac{\partial S_1}{\partial l} + \cdots\right) + \varepsilon F_1\left(L^* + \cdots, G^* + \cdots, l, -\right) \end{aligned} \tag{J.7.8}$$

Expanding F_0 and F_1 in powers of ε, we have

$$\begin{aligned} F_0^*(L^*) + \varepsilon F_1^*(L^*, G^*) + \cdots = F_0(L^*) + \varepsilon\frac{\partial F_1}{\partial L} + \\ \varepsilon F_1\left(L^* + \cdots, G^* + \cdots, l, -\right) + \cdots. \end{aligned} \tag{J.7.9}$$

Clearly,

$$F_0^*(L^*) = F_0(L^*) = \frac{\mu^2}{2L^{*2}}. \tag{J.7.10}$$

Then, equating terms of order ε,

$$F_1^*(L^*, G^*) = \frac{\partial F_0}{\partial L^*}\frac{\partial S_1}{\partial l} + F_1(L^*, G^*, l, -). \tag{J.7.11}$$

Now define the *secular* part of F_1 by

$$F_{1s} = \frac{1}{2\pi}\int_0^{2\pi} F_1(L, G, l, -)dl = \frac{1}{a^2\sqrt{1-e^2}}, \tag{J.7.12}$$

and the *periodic* part by

$$F_{1p} = F_1 - F_{1s} \tag{J.7.13}$$

Since the left hand side of (J.7.12) is independent of l, we set

$$\left.\begin{aligned} F_1^*(L^*, G^*) &= F_{1s}(L^*, G^*) = \frac{\mu^2}{L^{*3}G^*}, \\ \frac{\partial F_0}{\partial L^*}\frac{\partial S_1}{\partial l} &+ F_{1p}(L^*, G^*, l, -) = 0. \end{aligned}\right] \tag{J.7.14}$$

The new Hamiltonian has now been found (to the order ε) and so has the determining function, where

$$S_1 = \int F_{1p} dl = (v - l), \tag{J.7.15}$$

the equation of the center. With

$$F^*(L^*, G^*) = \frac{\mu^2}{2L^{*2}} + \varepsilon \frac{\mu^2}{L^{*3}G^*}, \tag{J.7.16}$$

Hamilton's equations have the solution $L^* = \text{const.}$, $G^* = \text{const.}$,

$$l^* = l_0 + \left(\frac{\mu^2}{L^{*3}} + \varepsilon \frac{3\mu^2}{L^{*4}G^*} \right) t, \quad g^* = g_0 + \varepsilon \frac{\mu^2}{L^{*3}G^{*2}} t, \tag{J.7.17}$$

giving the secular perturbations. Then, from

$$S = L^* l + G^* g + \varepsilon(v - l) \tag{J.7.18}$$

we have

$$L = L^* + \varepsilon \left(\frac{\partial v}{\partial l} - 1 \right), \; G = G^*, \; l^* = l + \varepsilon \frac{\partial v}{\partial L}, \; g^* = g + \frac{\partial v}{\partial G}, \tag{J.7.19}$$

to the first order in ε. $\frac{\partial v}{\partial L}$ can be found at once from the integral expression, (J.7.16), for S_1. The remaining derivatives can be found by applying the chain rule using formulas for elliptic motion. This is done in Ref. 19, where the approach just used, is applied to the problem of the motion of an artificial satellite. This method was proposed by Poincaré, and applied to orbits of minor planets by von Zeipel; it is often referred to as the "von Zeipel method."

Bibliography and References

Chapter 1

There are many first-rate introductory texts in astronomy available, and a perusal of some of these will provide pleasure as well as information. For more detailed information on the astronomical background pertinent to the dynamics of the solar system and space flight, see:

1. Baker, R.M.L., and Makemson, M.W. *An Introduction to Astrodynamics.* New York: Academic Press, 1967.

For information on spherical astronomy, see:

2. Smart, W.M. *Textbook on Spherical Astronomy.* Cambridge, England: Cambridge University Press, 1956.
3. Woolard, E.W. and Clemence, G.M. *Spherical Astronomy.* New York: Academic Press, 1966.

For a gold mine of practical information, see:

4. *Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac.* London: Her Majesty's Stationery Office. First published in 1960, but updated in more recent printings.

Also refer to:

5. *The Astronomical Almanac* for the current year, issued by the Nautical Almanac Offices in the U.S. and Great Britain, and available through Washington: U.S. Government Printing Office, or London: Her Majesty's Stationery Office.

Chapter 2

Many texts include introductions to vectors. But beware of those that define vectorial properties in terms of their components. The references quoted here are:

6. Pars, L.A. *Introduction to Dynamics.* Cambridge, England: Cambridge University Press, 1953.