Solar System Dynamics

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Abstract

A fly-by to the basic dynamics of our Solar System.

Outline

- Forces
- Field of an isolated body
- Two–body problem
- N–body problem
- Planetary theory
- Resonances
- Three–body problem
- Appendix

Forces

Forces

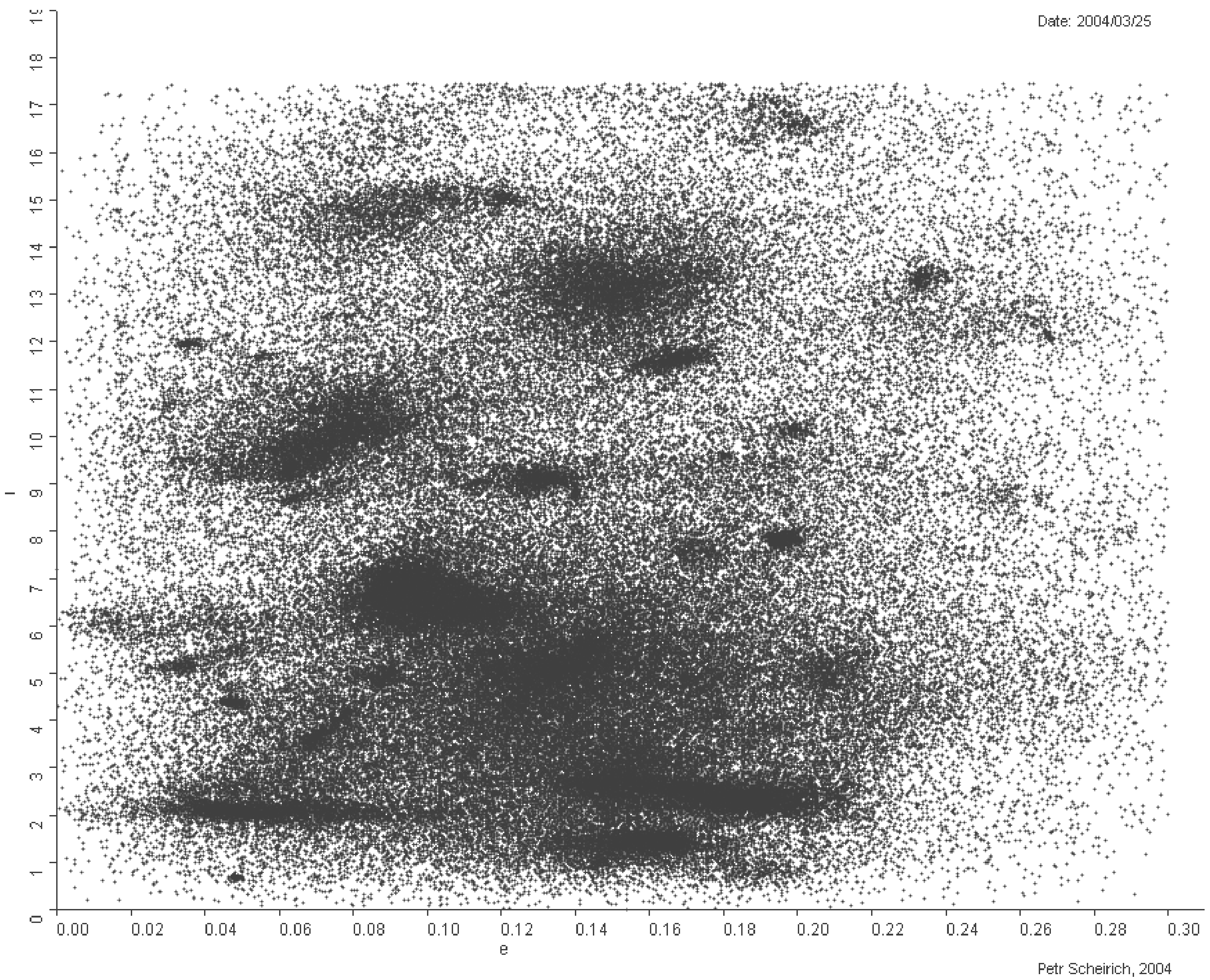
Generated by

- gravity (Newtonian and relativistic) due to Sun, planets, satellites, asteroids.

  Model: N point masses + perturbations due to oblateness.

- solar radiation
- medium: solar wind, gas drag.
- magnetic fields: Lorentz forces.
- collisions!

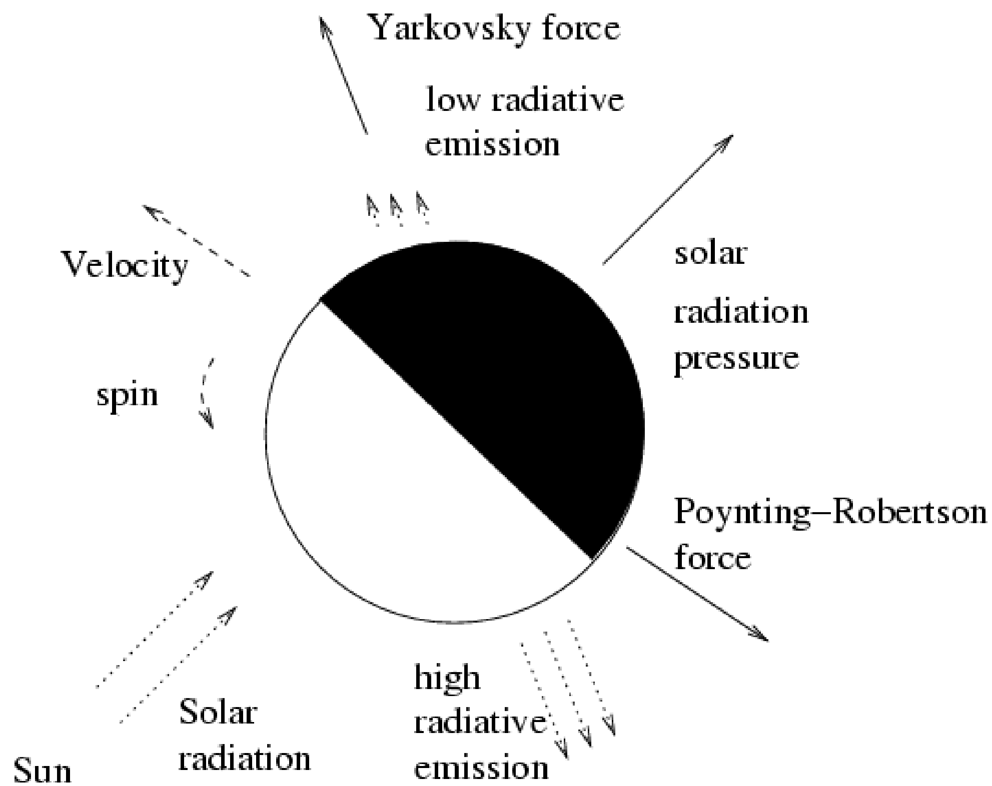
In the space of orbital elements the asteroid population shows several concentrations: they are produced by several fragments generated after a catastrophic collision. All them have very similar orbital elements and constitute a **FAMILY**.

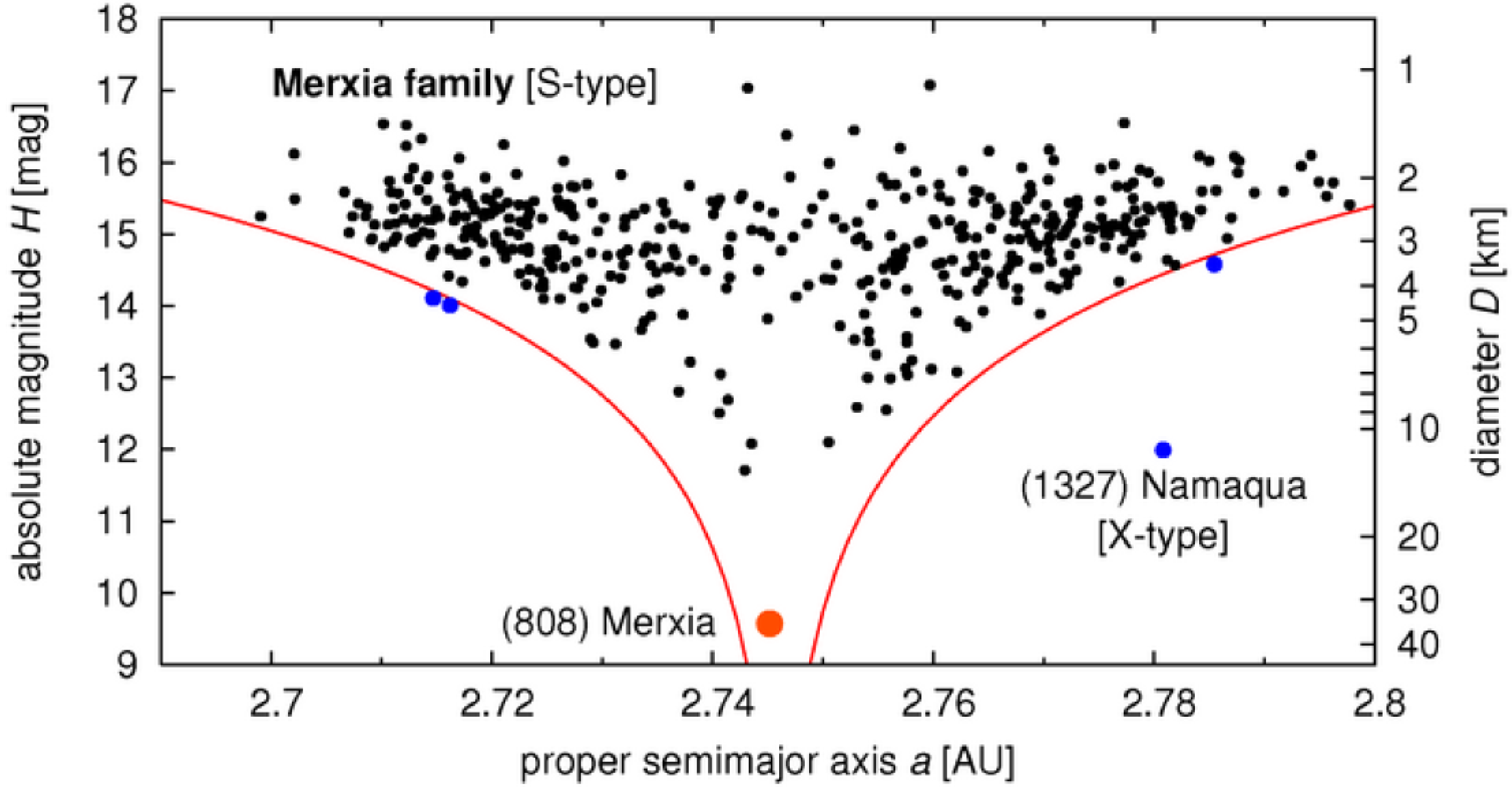


ECCENTRICITY versus INCLINATION

Solar radiation generates several dynamical effects:

- radiation pressure (μm): in the direction of the radiation
- Poynting-Robertson drag (cm): (Doppler) opposite to velocity generates migration to the Sun
- Yarkovsky effect (from m to km): (thermal inertia) depending on rotation generates migration to or from the Sun
- sublimation in comets (ask to some experts here in this room....)





(taken from Broz et al. 2005)

After a collision a family is generated: the smaller fragments (higher magnitude) are the most affected by Yarkovsky (so, the most dispersed). This effect can help us in the **determination of the age of the family**.

For most of the solar system's bodies we can follow the scheme:

total Force model

$$=$$

point mass solar Newtonian gravity

$$+$$

several small perturbations

An isolated body

The field generated by the Sun

Consider a spherical Sun with density $\rho(r)$. Consider the origin at the Sun's center and a particle located at $\vec{r}$:

$$\ddot{\vec{r}} = -GM_{\odot}\frac{\vec{r}}{r^3} = -\nabla V$$

$$V = -\frac{GM_{\odot}}{r} \qquad \text{"potential" (the "energy" is conserved)}$$

This second order equation admits the following integrals:

constant "angular momentum" $\quad \vec{r} \wedge \dot{\vec{r}} = \vec{h} \quad$ (motion is planar)

constant "energy" $\quad \mathcal{E} = \frac{v^2}{2} - \frac{GM_{\odot}}{r}$

In polar (r, f) coordinates the trajectory satisfies:

$$r(f) = \frac{a(1-e^2)}{1+e\cos f}$$

which is a conic with semimajor axis a and eccentricity e (first law of Kepler). Defining $\mu = GM_\odot$ we can show that

$$\mathcal{E} = -\frac{\mu}{2a}$$

and

$$h^2 = \mu a(1 - e^2)$$

$\mathcal{E}$	$1/a$	conic	e
-	$+$	ellipse	$0 \leq e \leq 1$
$+$	-	hyperbola	$e \geq 1$
0	0	parabola	1

A very useful formula deduced from equation of energy is:

$$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

from which we obtain two very important velocities: the **circular** $(r \equiv a)$ velocity

$$v_c^2 = \mu\left(\frac{2}{r = a} - \frac{1}{a}\right) \Longrightarrow v_c = \sqrt{\frac{\mu}{r}}$$

and the **escape** $(a = \infty)$ velocity

$$v_e^2 = \mu \left(\frac{2}{r} - \frac{1}{\infty} \right) \Longrightarrow v_e = \sqrt{2} v_c$$

A particle located at a distance r from the Sun will have an hyperbolic orbit if its velocity verifies $v > \sqrt{2} v_c$, no matter the direction is.

Another important velocity is the velocity **at infinity** $(r = \infty)$, the one that the particle has when the term $2\mu/r$ can be neglected in front of v^2

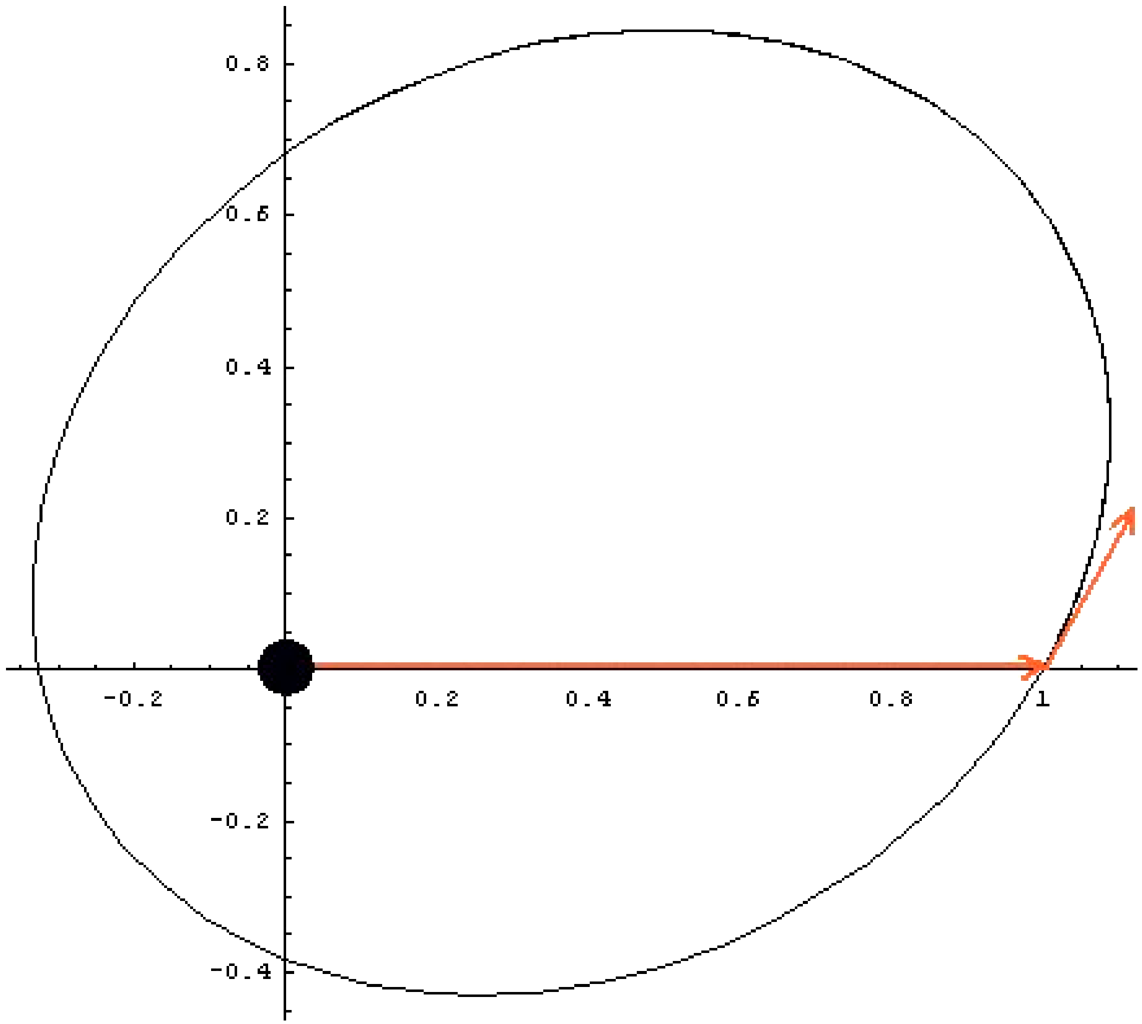
$$v_\infty^2 = \mu \left(\frac{2}{\infty} - \frac{1}{a} \right) = -\frac{\mu}{a}$$

v_∞ does not exist for ellipses and $v_\infty = 0$ for parabolas.

Semimajor axis is determined by the modulus of r and v, but the eccentricity is defined by

$$\mu a(1 - e^2) = h^2 = |\vec{r} \wedge \vec{v}|^2$$

so it depends also on the angle between $\vec{r}$ and $\vec{v}$



Second and third laws of Kepler

By definition

$$h = |\vec{r} \wedge \vec{v}| = r \cdot r\frac{df}{dt} = 2\frac{dA}{dt} = \text{ constant}$$

then **the areolar velocity is constant** (second law of Kepler). In the particular case of an ellipse:

$$h = \sqrt{\mu a(1-e^2)} = 2\frac{\text{area ellipse}}{\text{orbital period}} = 2\frac{\pi ab}{P}$$

operating, the mean angular velocity called **mean motion** is:

$$n = \frac{2\pi}{P} = \sqrt{\frac{\mu}{a^3}}$$

which is the third law given by Kepler. The orbital period P only depends on the semimajor axis.

An artificial satellite orbiting the Earth at $a = 42160$kms has an orbital period exactly equal to the rotation period of the Earth and is called an "geosynchronous satellite".

Orbital elements

The second order differential equation

$$\ddot{\vec{r}} = -\mu \frac{\vec{r}}{r^3}$$

has 6 independent integrals of motion $(a, e, i, \varpi, \Omega, T)$ determined by the initial conditions $(x_o, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o)$. These integrals of motion are called orbital elements:

- a and e define the shape
- i, ϖ and Ω define the spatial position of the orbit
- T defines the instant of the passage by the pericenter

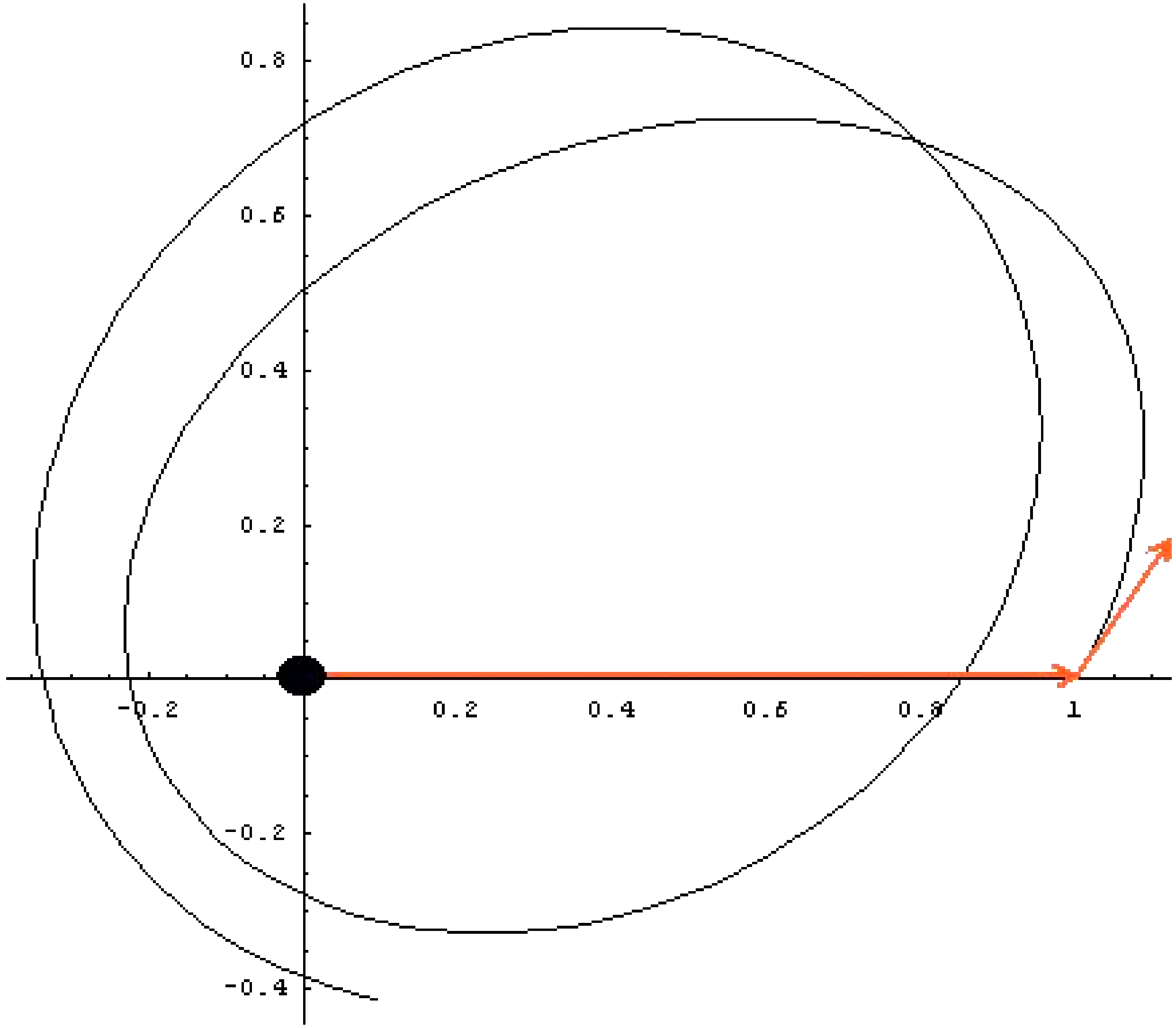
and they are constant.

A non spherical body

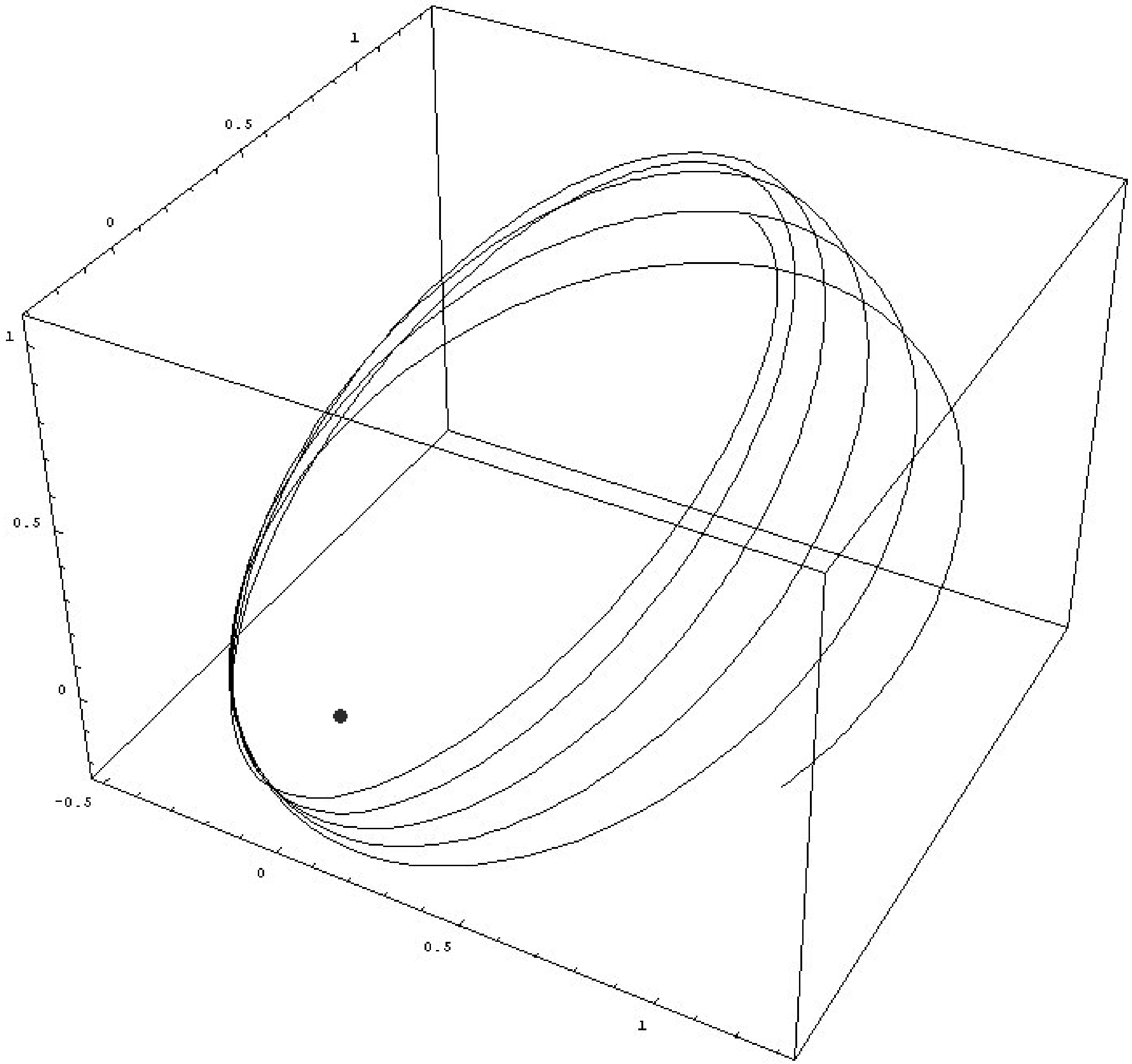
If the body is not spherical

$$\ddot{\vec{r}} = -\mu\frac{\vec{r}}{r^3} + \nabla R(\vec{r}, t)$$

where R is called DISTURBING FUNCTION. Considering the same initial $\vec{r}$ and $\vec{v}$ the resulting motion is a **perturbed** conic, that means, varying with time.



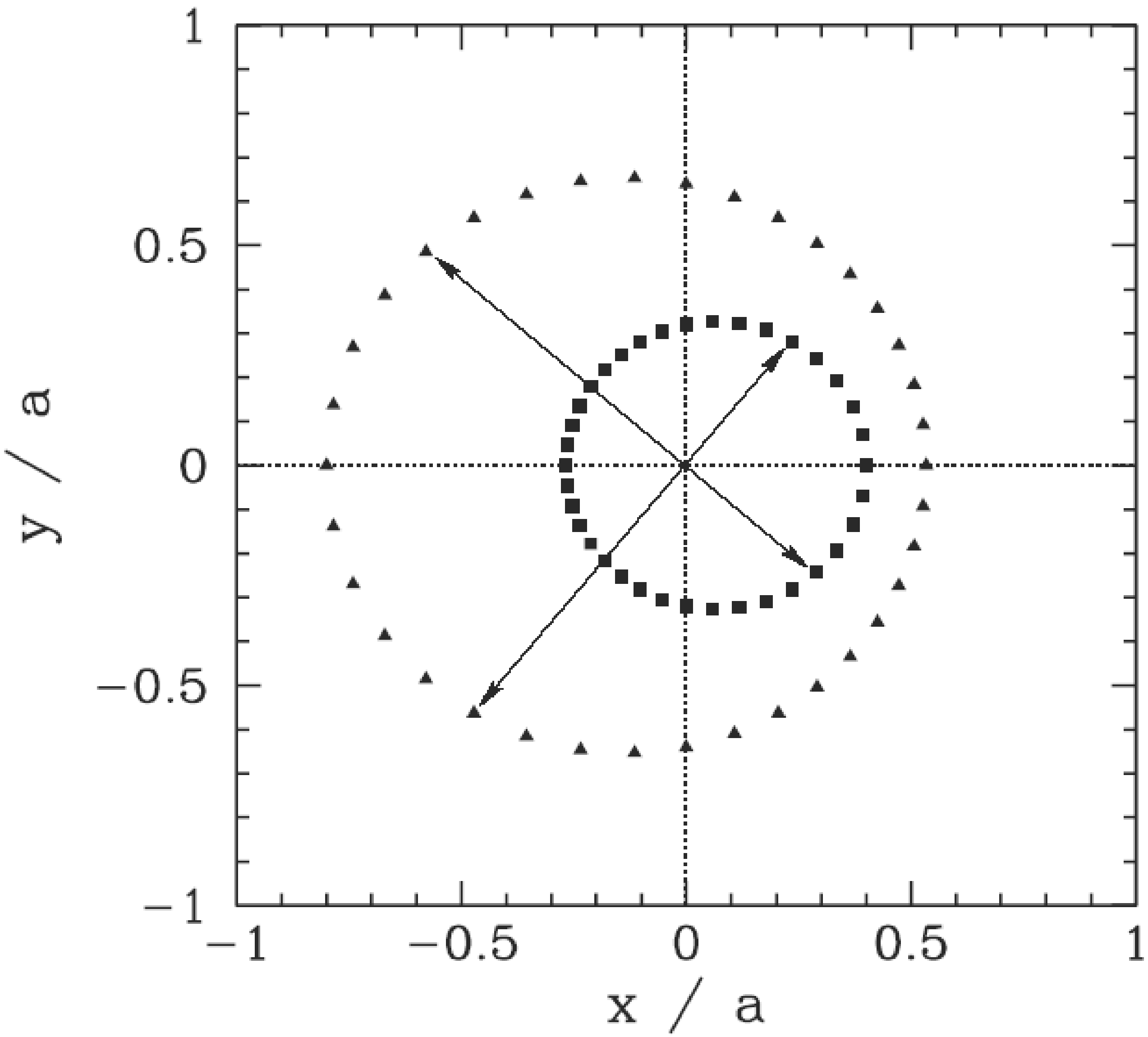
There is an "orbital evolution" given by $(a(t), e(t), i(t), \varpi(t), \Omega(t), T(t))$. For a given EPOCH t_o (for example now, JD2454305.25) we have an instantaneous "osculating" orbit given by the set $(a(t_o), e(t_o), i(t_o), \varpi(t_o), \Omega(t_o), T(t_o))$.



In general, the perturbed orbit is not planar.

Two–Body Problem

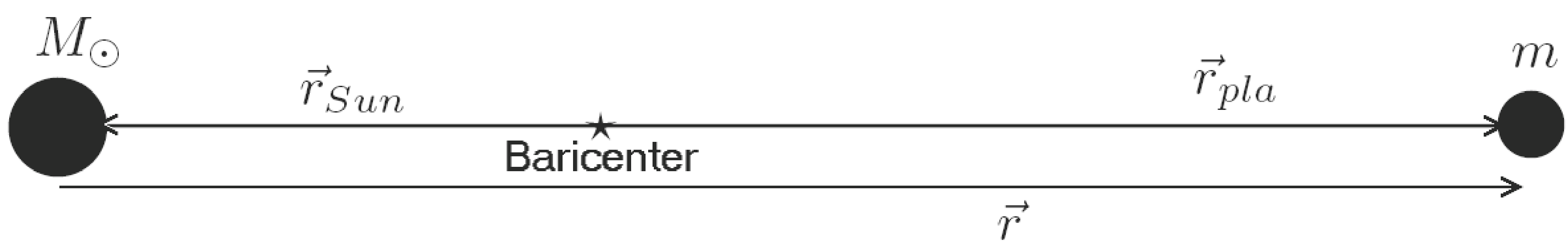
The bodies accelerate each other



(from Fitzpatrick)

Sun + Planet

The planet m accelerates the Sun, then it is not more inertial and we cannot apply Newtonian laws directly with origin in the Sun. We can take the baricenter as an inertial origin and we can write the equations of motion related to this frame:



$$\vec{F}_{Sun} = M_\odot \ddot{\vec{r}}_{Sun} = +GM_\odot m \frac{\vec{r}}{r^3}$$

$$\vec{F}_{pla} = m \ddot{\vec{r}}_{pla} = -GM_\odot m \frac{\vec{r}}{r^3}$$

being $\vec{r} = \vec{r}_{pla} - \vec{r}_{Sun}$ the position vector of the planet relative to the Sun.

Operating:

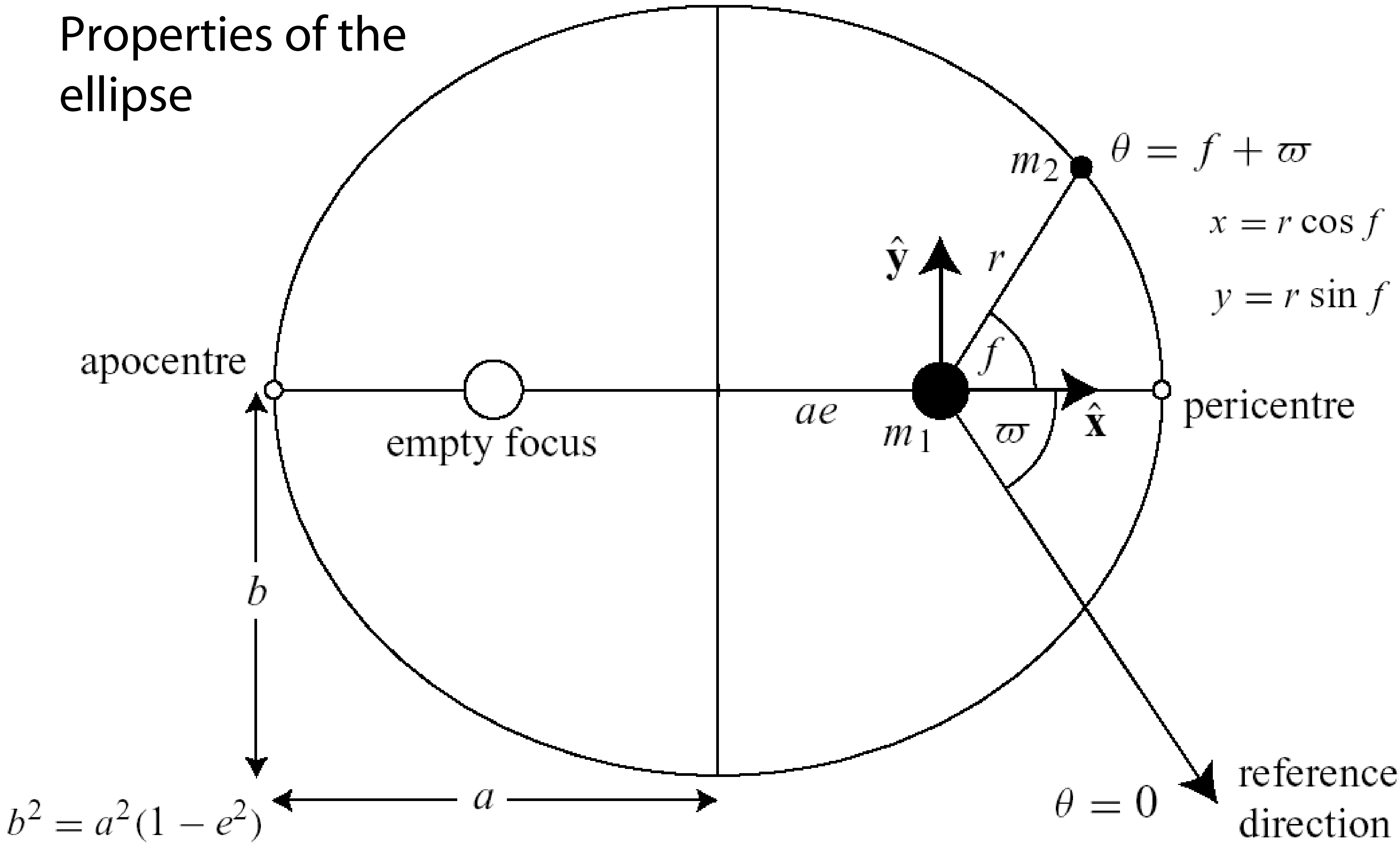
$$\ddot{\vec{r}} = \ddot{\vec{r}}_{pla} - \ddot{\vec{r}}_{Sun} = \frac{\vec{F}_{pla}}{m} - \frac{\vec{F}_{Sun}}{M_\odot} = -G(M_\odot + m)\frac{\vec{r}}{r^3}$$

$$\Longrightarrow \ddot{\vec{r}} + \mu\frac{\vec{r}}{r^3} = 0$$

being the same equation of the conic but now with $\mu = G(M_\odot + m)$.

The motion of a planet of mass m around the Sun is equal to the motion of a massless particle around a star with mass $(M_\odot + m)$.

The shape of the orbit



(from Carl Murray)

pericentre: $r(f = 0) = a(1 - e) = q$, apocentre: $r(f = \pi) = a(1 + e) = Q$

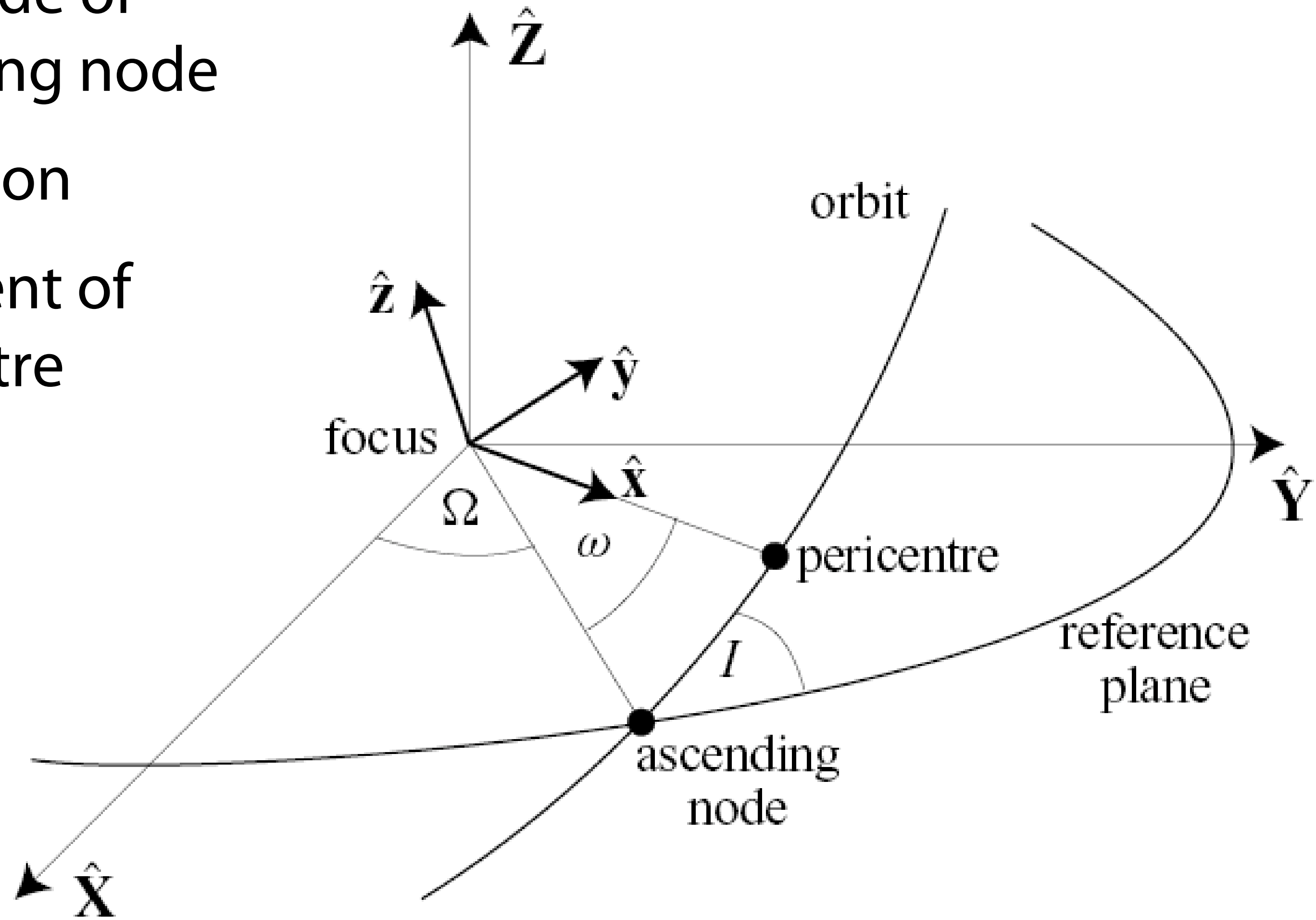
Orbit in space

□ Longitude of ascending node

I Inclination

□ Argument of pericentre

$$\varpi = \Omega + \omega$$



(from Carl Murray)

The elements Ω, ω, i need some reference plane and some reference direction, for example, the ecliptic and Aries corresponding to 2000.0.

N–Body Problem

General N body problem

We start with N second order differential equations in vectorial form:

$$\ddot{\vec{r}}_i = -G \sum_{j=1}^{N} m_j \frac{\vec{r}_j - \vec{r}_i}{r_{ji}^3}$$

That system can be transformed to a system of $2N$ first order differential equations. According to a theorem (from differential equations) we need $2N \times 3$ integrals of motion in order to resolve the system. Unfortunately there exist only 10 integrals, then for $N > 2$ the N body problem has not analytical solution. We can instead obtain the numerical solutions $\vec{r}_i(t)$ which are unique.

10 integrals of motion

An isolated system verifies constant momentum $\overrightarrow{P}$

$$\overrightarrow{P} = \sum_{j=1}^{N} m_j \overrightarrow{v}_j = \overrightarrow{v}_{bar} \sum_{j=1}^{N} m_j = \text{ constant}$$

so its barycenter moves with constant velocity, it is inertial and we can take it as the origin of coordinates. Then:

$$\overrightarrow{v}_{bar} = \text{ (linear comb. of } \vec{v}_j) = 0 \qquad \text{(3 constants of motion)}$$

and also

$$\overrightarrow{r}_{bar} = \text{ (linear comb. of } \vec{r}_j) = 0 \qquad \text{(3 constants of motion)}$$

Angular momentum is also constant:

$$\overrightarrow{L} = \sum_{j=1}^{N} m_j \overrightarrow{r}_j \times \overrightarrow{v}_j = \text{ constant} \qquad \text{(3 more constants)}$$

The fact that $\overrightarrow{L}$ is constant means there is a preferred spatial direction. The plane perpendicular to this direction is called the **fundamental** or **invariable plane**.

Total energy:

$$\mathcal{E} = T + \mathcal{E}_p = \frac{1}{2}\sum_{j=1}^{N} m_j v_j^2 - G\sum_{j=1}^{N}\sum_{k=1}^{j-1}\frac{m_k m_j}{r_{kj}} = \text{ constant} \qquad \text{(1 more constant)}$$

These 10 integrals are not enough to resolve analytically the system of N vectorial second order differential equations

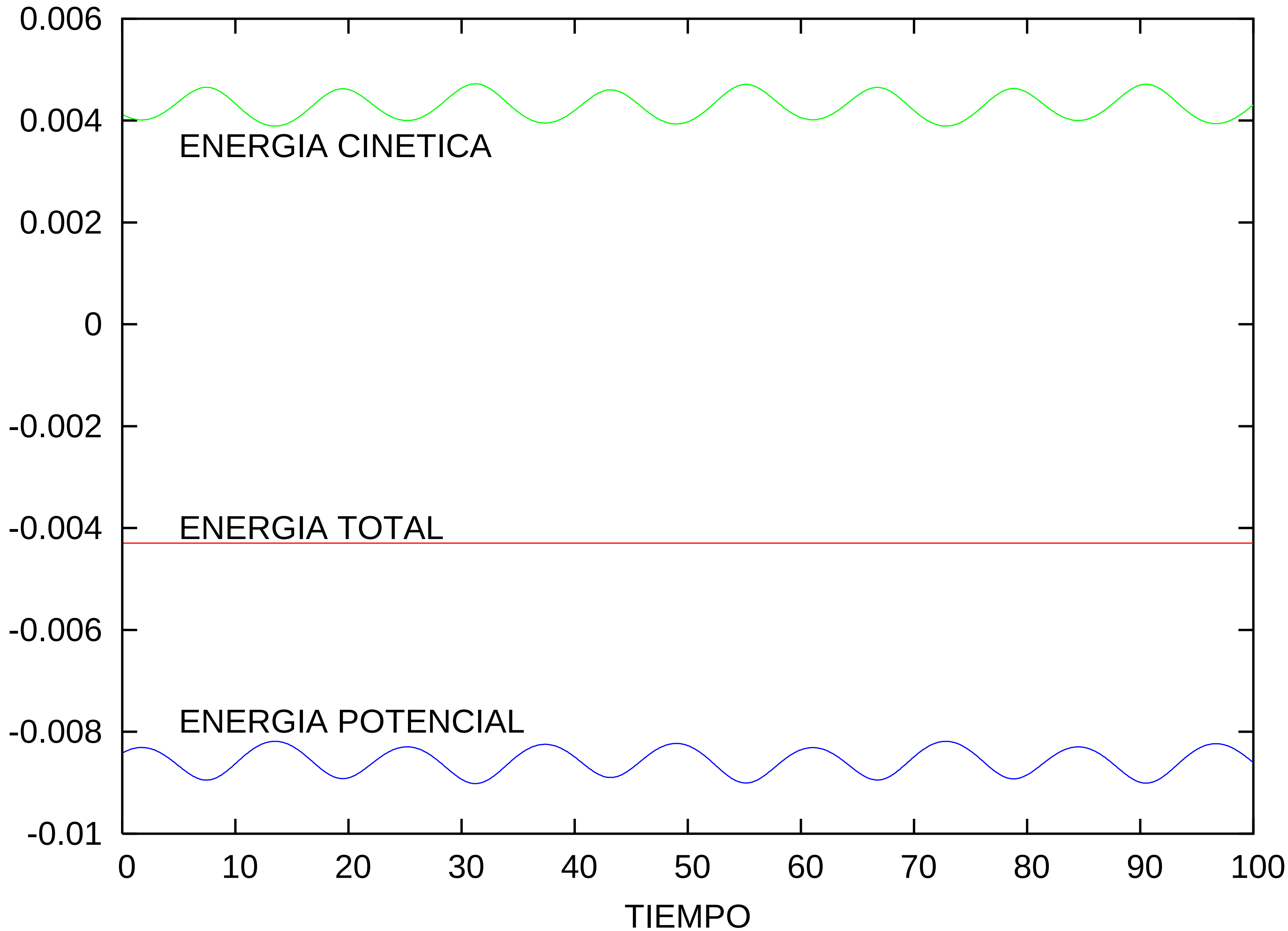
$$\ddot{\vec{r}}_i = -G\sum_{j=1}^{N} m_j \frac{\vec{r}_j - \vec{r}_i}{r_{ji}^3}$$

but we can solve it numerically using appropriate algorithms called "numerical integrators": MERCURY, SWIFT, HNBody, OrbFit, EVORB, and many others.

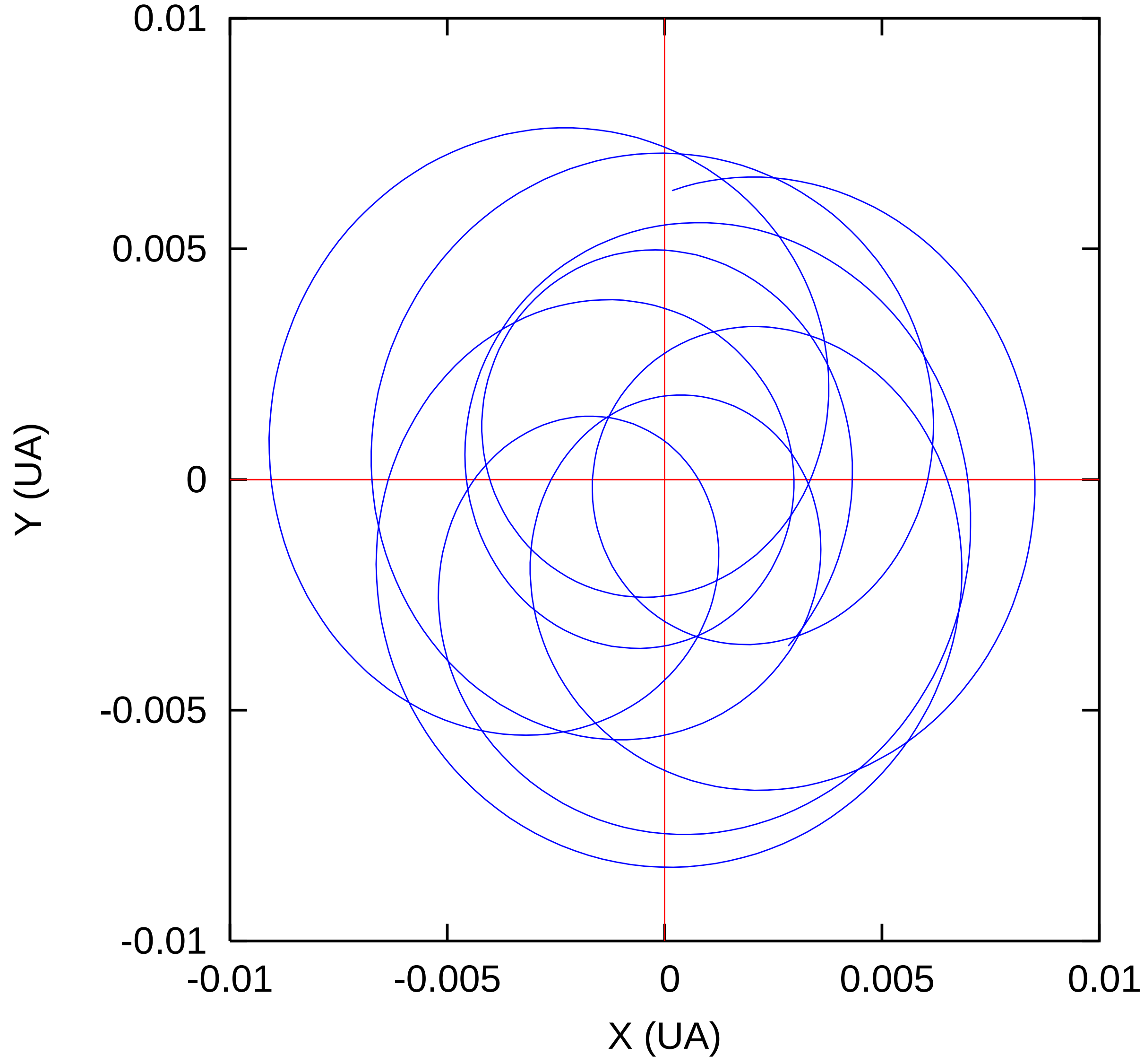
For each body we obtain

$$x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)$$

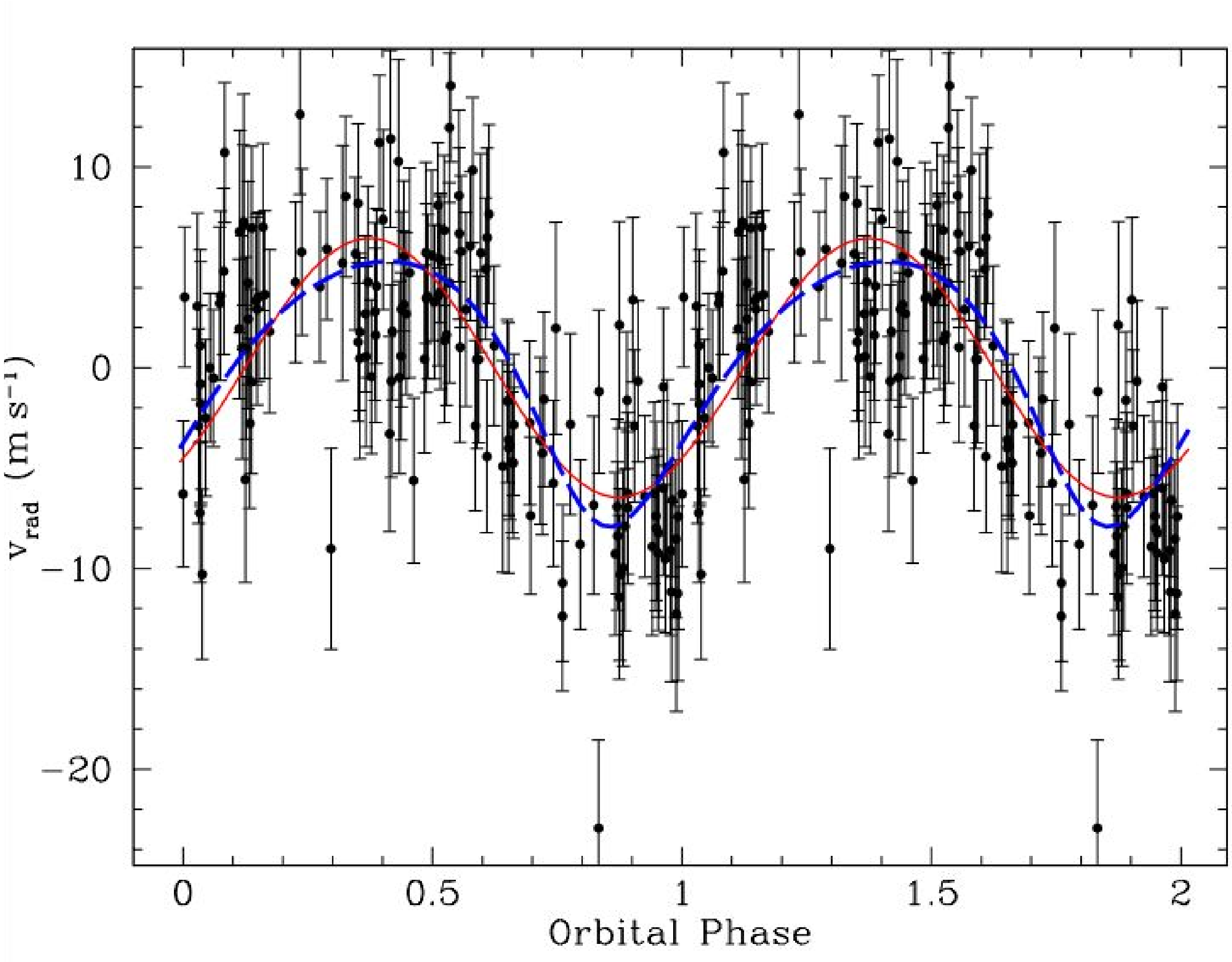
Time evolution of the Solar System's energy



100 years of motion of the Sun around the baricenter



Extraterrestrial civilizations could detect this motion (don't panic).



(from ... somewhere...)

Planetary theory

Planetary problem: N planets + Sun

The planetary problem can be considered as an (N+1) body problem with a dominating mass (the Sun). We can transform the (N+1) vectorial equations relative to the baricenter to a sistem of N heliocentric equations:

$$\ddot{\vec{r}}_i = -\mu_i \frac{\vec{r}_i}{r_i^3} + G \sum_{j \neq i}^{N} m_j \left(\frac{\vec{r}_j - \vec{r}_i}{r_{ji}^3} - \frac{\vec{r}_j}{r_j^3} \right)$$

$$\mu_i = G(M_\odot + m_i)$$

$$\boxed{\text{heliocentric acceleration} = \text{two-body} + \text{direct pert. by } m_j + \text{indirect pert. by } m_j}$$

$$\boxed{\text{planetary motion} = \text{two-body problem} + \text{small perturbations}}$$

Instead of solving the rectangular heliocentric quick varying coordinates (timescale of days)

$$x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)$$

we can solve the heliocentric slow varying parameters (timescale of centuries)

$$a(t), e(t), i(t), \varpi(t), \Omega(t), T(t)$$

Perturbation Theory

For a given planet it is possible to write the equation of motion in the form

$$\ddot{\vec{r}} + \mu \frac{\vec{r}}{r^3} = \nabla R(\vec{r}_1, \ldots, \vec{r}_N)$$

where R is the Disturbing Function. It is possible to transform this equation in another very different form due to Lagrange (+ Euler + Laplace):

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \lambda} \\
\frac{de}{dt} &= \cdots \\
\frac{di}{dt} &= \cdots \\
\frac{d\varpi}{dt} &= \cdots \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{dT}{dt} &= \cdots
\end{aligned}$$

6 equations for each planet where R is a very unfriendly function

$$R = R(a_1, ..., a_N, e_1, ..., e_N, i_1, ..., i_N, \varpi_1, ..., \varpi_N, \Omega_1, ..., \Omega_N, \lambda_1, ..., \lambda_N)$$

In general it is a function depending on $6 \times N$ variables and to obtain that expression for R it was not a trivial issue:

$$R = \sum F(a, e, i) \cos[S(\lambda, \varpi, \Omega)]$$

where functions $S(\lambda, \varpi, \Omega)$ are linear combinations of λ, ϖ, Ω. The λ are quick varying angles, on the contrary ϖ, Ω are slow varying angles. Then:

$$R = R_{SP}(a, e, i, \varpi, \Omega, \lambda) + R_{LP}(a, e, i, \varpi, \Omega)$$

The short period terms usually cancellate (please theoreticians close your eyes) and we can assume

$$R \simeq R_{LP}(a, e, i, \varpi, \Omega)$$

this part of the disturbing function is the responsible for the long term **secular evolution** of the system.

Taking $R \simeq R_{LP}$ the first of the Lagrange's planetary equations becomes:

$$\frac{da}{dt} \simeq \frac{2}{na}\frac{\partial R_{LP}}{\partial \lambda} = 0$$

then the semimajor axis of the planets do not change with time...

the planetary system do not shrinks nor expands!!!

That was a very impacting result of the XVIII century due to Euler, Lagrange and Laplace. It is also possible to show that e and i do not grow systematically but oscillate.

the band

Euler

Laplace

Lagrange

Jacobi

LeVerrier

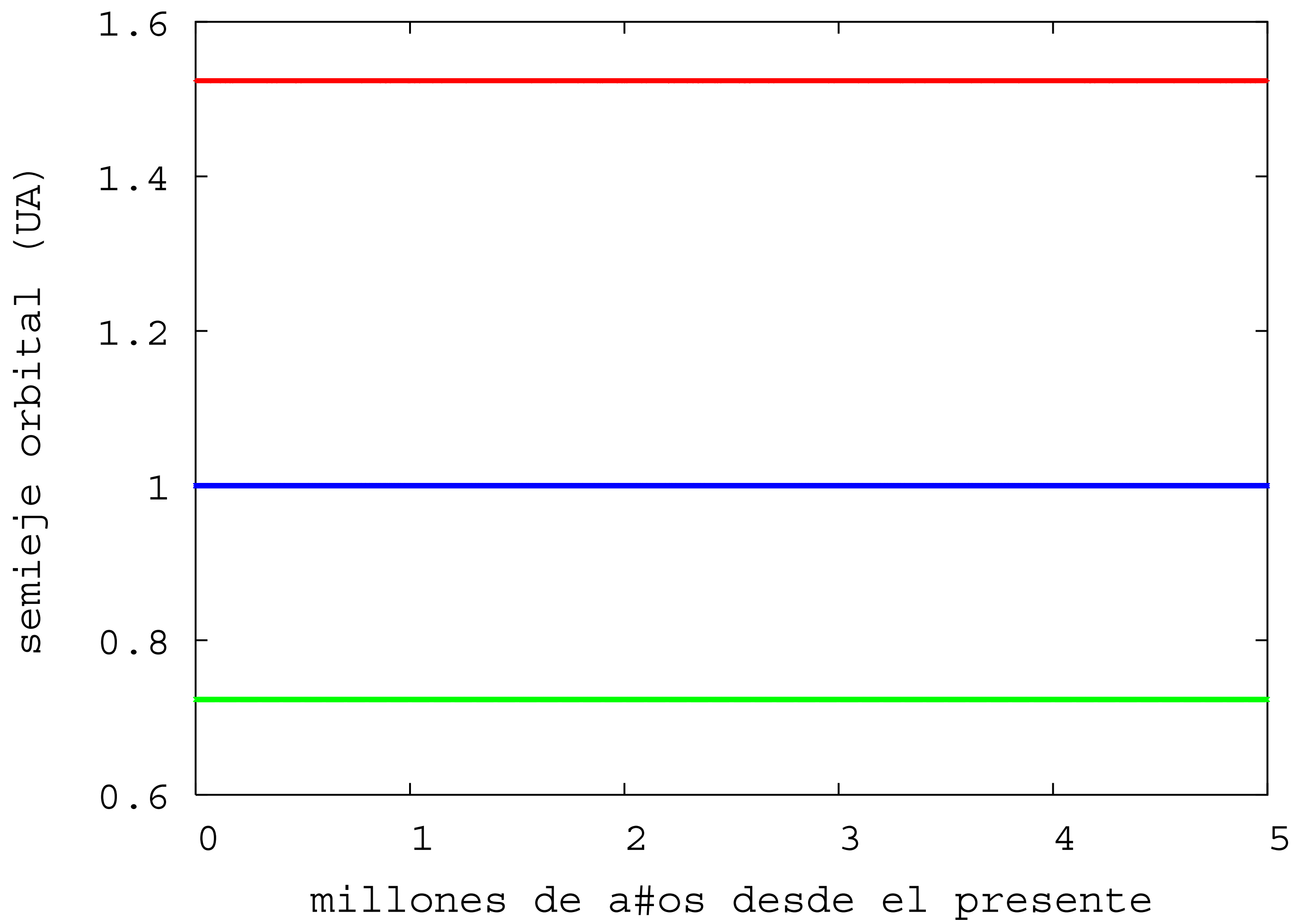
Hamilton

Birkoff

Poincaré

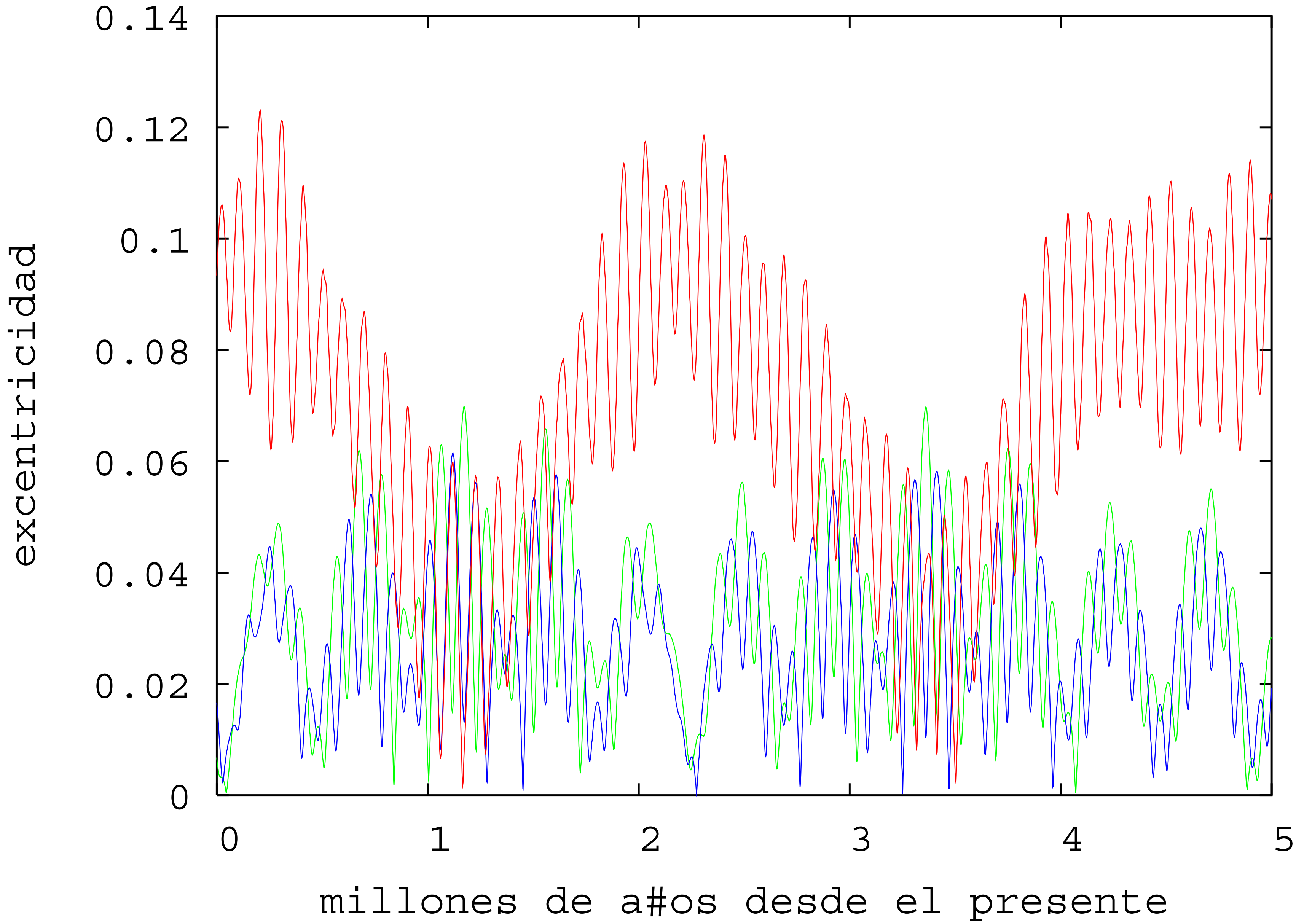
(from Carl Murray)

Venus, Earth and Mars in the next 5 Myrs: $a(t)$



Results from the numerical integration of the exact newtonian equation of motion of the Solar System. Congratulations to Euler-Laplace-Lagrange!!!

Venus, Earth and Mars in the next 5 Myrs: $e(t)$



Fundamental Frequencies and Chaos

If planetary orbits are not very close and with small eccentricity it is possible to show that the system will oscillate with $3N$ -almost well defined- Fundamental Frequencies:

- N high frequencies are associated with the orbital periods (years) and generate small oscillations in a
- N low frequencies are associated with the "precession" of the perihelia and generate oscillations in e $(10^5 - 10^6$ years)
- N low frequencies are associated with the motion of the orbital nodes and generate oscillations in i $(10^5 - 10^6$ years)

The first set of N frequencies is not related to relevant variations in the orbital elements so usually are ignored. The last two sets of N frequencies are known as the **fundamental frequencies of the Solar System**.

If these frequencies are well defined (that means, constant) the system is quasi-regular and stable. If these frequencies are not well defined (varying with time) the system is chaotic and the chaotic behavior will be appreciable after some timescale.

Our Solar System has not fixed fundamental frequencies then ..... is chaotic.

So...

the planetary system is stable and chaotic ...

- According to the perturbation theory (semimajor axes are constant) the system is STABLE.
- According to the N-body problem the future of the system is DETERMINED (only one solution exists) but CHAOTIC (hard to predict).
- According to modern numerical integrations the planetary system is under STABLE CHAOS: we can predict reasonably well the orbital evolution but not the exact position of the planets in their orbits.

Numerical integrations versus theory

Nowadays theoretical analysis is used not just to obtain analytical solutions but to provide theoretical explanations to the very precise solutions obtained with the numerical integrators.

Everybody can obtain a precise numerical solution of a dynamical problem but only with the understanding of the theory we can explain the results obtained.

Resonances

Resonances

They happen when a simple commensurability exists between some fundamental frequencies of the system (orbital periods, rotational periods, perihelion motion, node motion)

- orbit-orbit (mean motion resonances)
  - asteroids with Jupiter (3:2, 2:1, 1:1)
  - TNOs with Neptune
  - Pluto-Neptune (2:3)
  - quasi resonance Jupiter-Saturn (5:2) and Uranus-Neptune (2:1)
  - galilean satellites of Jupiter, Uranus satellites, rings
  - spacecrafts
- spin-orbit
  - Earth-Moon (1:1)
  - Sun-Mercury (1:2)
  - Pluto-Charon (1:1)
- secular resonances: orbital plane involved (ϖ, Ω). Fatal destiny: collision with the Sun.

Is the resonant motion a common dynamical state?

For example, mean motion resonances with a given planet are located at

$$n \cdot N_1 \simeq n_p \cdot N_2 \quad \text{or} \quad a \simeq a_p \left(\frac{N_1}{N_2}\right)^{2/3}$$

with N_1, N_2 arbitrary integers. Then, resonances occur at very precise positions. Is it probable that an object be located at the exact position of the resonance?

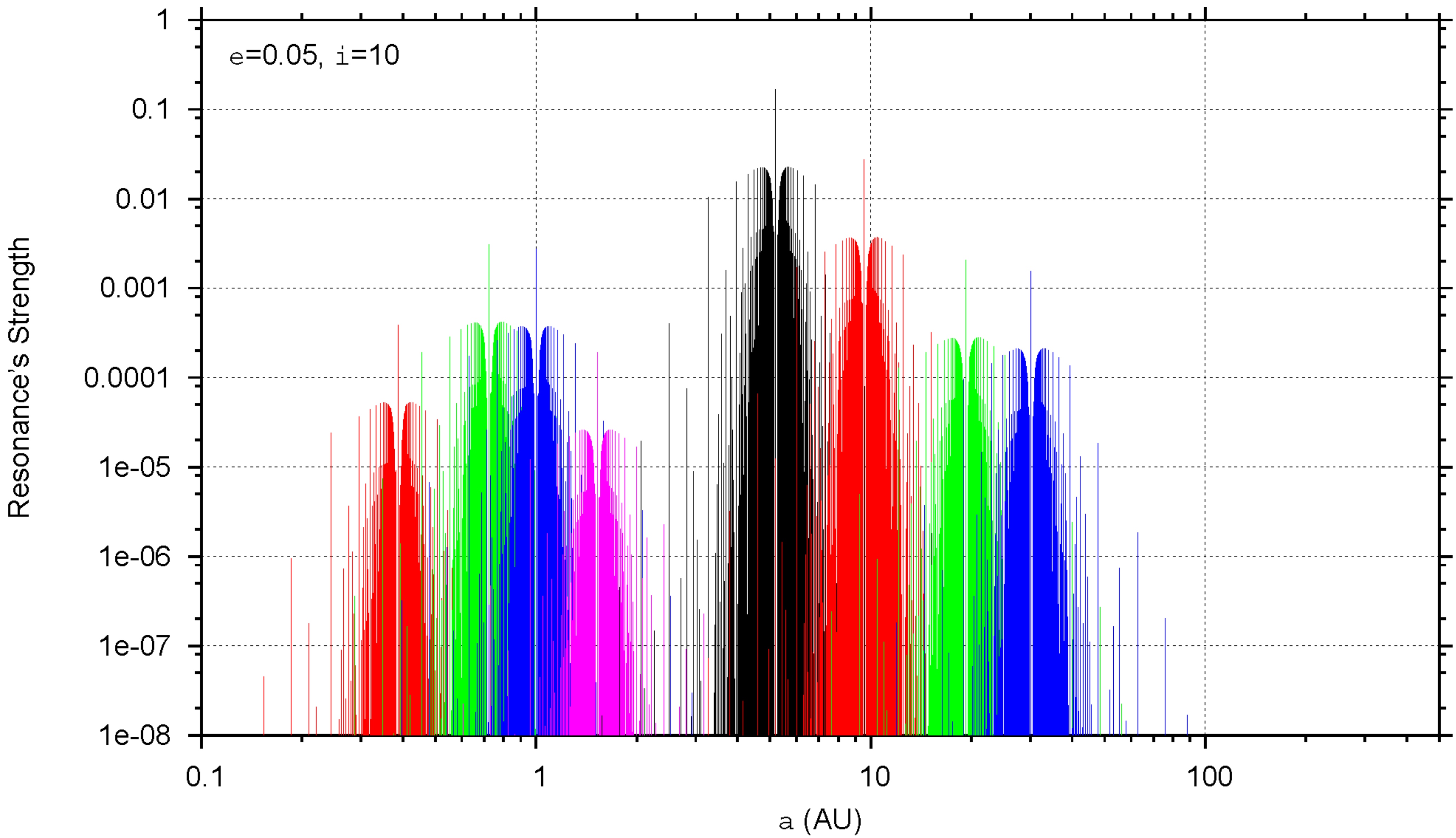
Yes

Why?

At least three reasons:

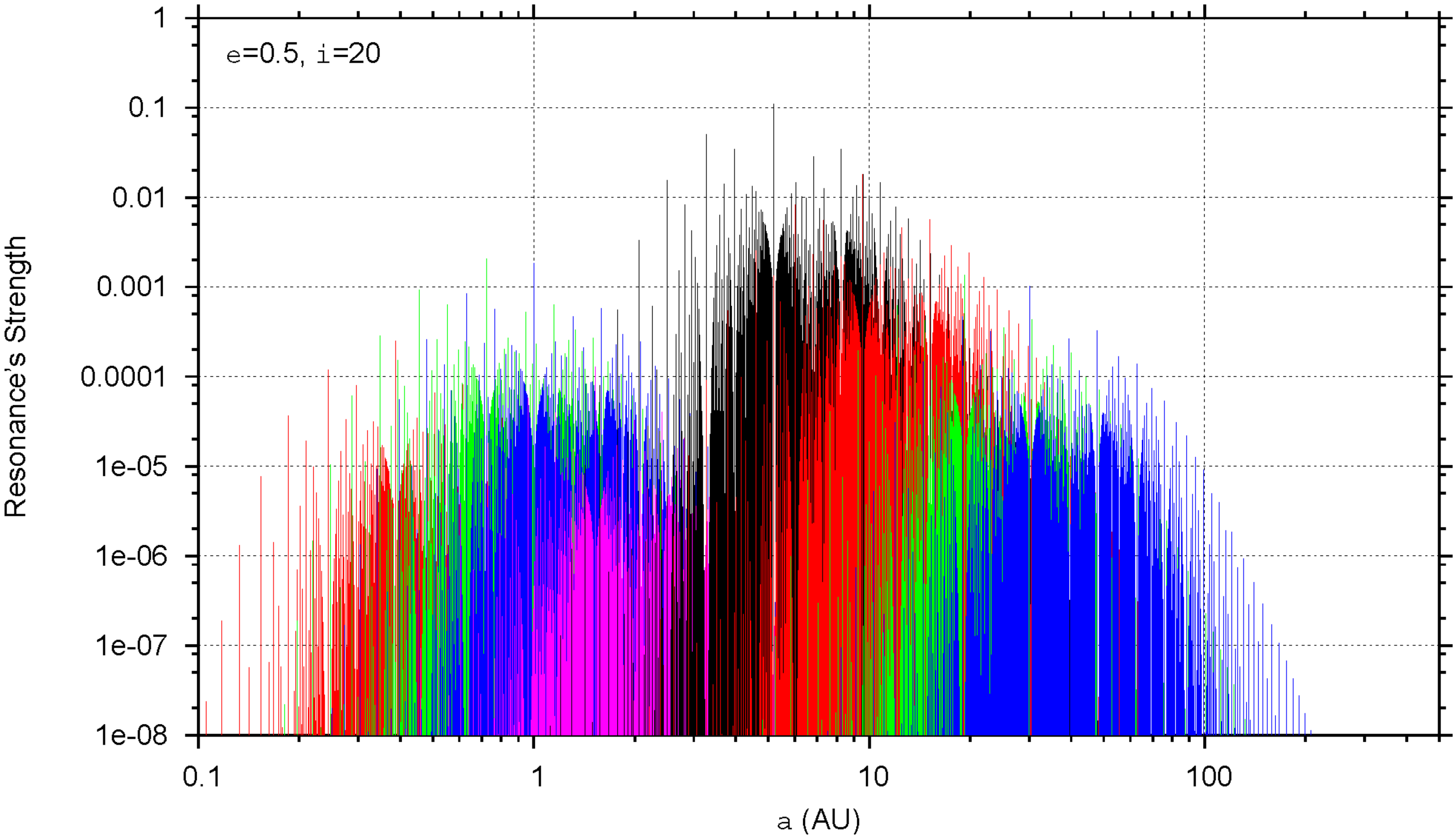
- there are several, several resonances
- resonances have some **strength** and **stickiness**, they can "attract" trajectories to them
- there are mechanisms (like Yarkovsky, tides, gas drag) that drive the objects to the resonances and there is a chance to be captured by them

Resonances everywhere



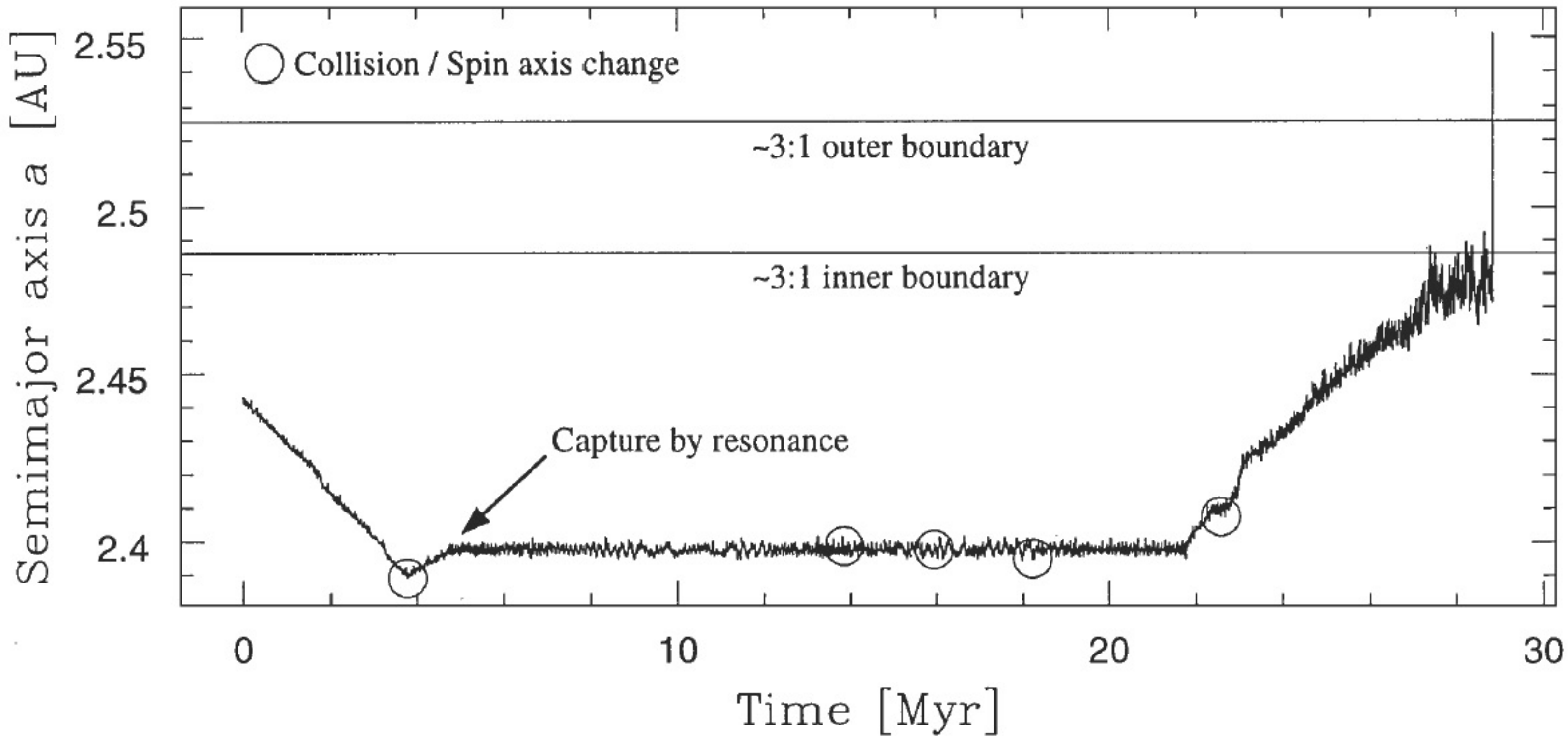
Strengths for low eccentricity orbits.

Resonances everywhere



Strengths for high eccentricity orbits.

Resonance capture due to Yarkovsky

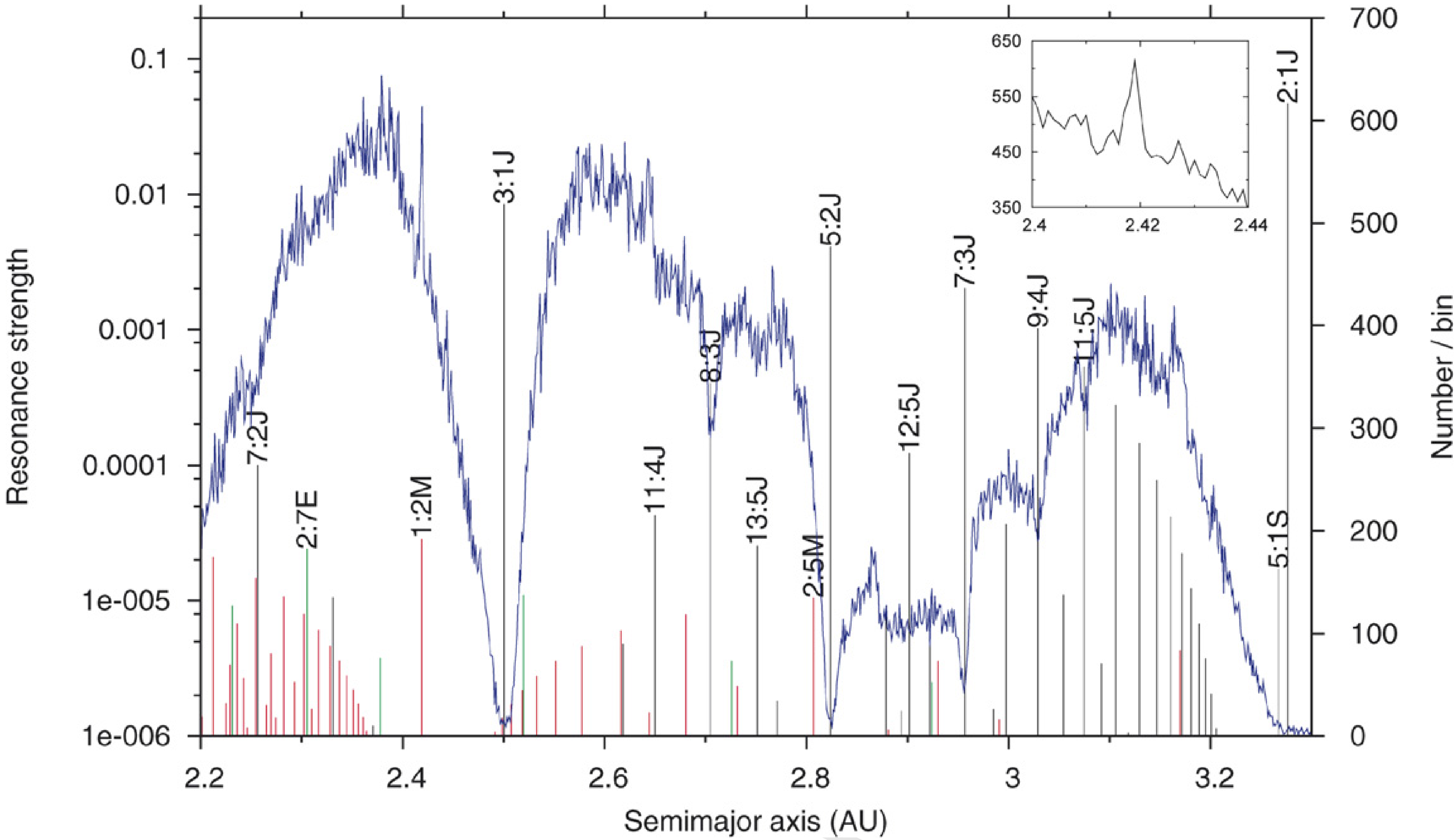


(Bottke et al. 2000)

Then, high eccentricity orbits (and sometimes high inclination orbits) are usually in resonance. A notable example is the orbital evolution of the population of NEOs, comets, TNOs and SDOs.

The resonant motion only appears after several orbital revolutions, it is not an instantaneous effect.

The Main Belt of asteroids is shaped by Jupiter



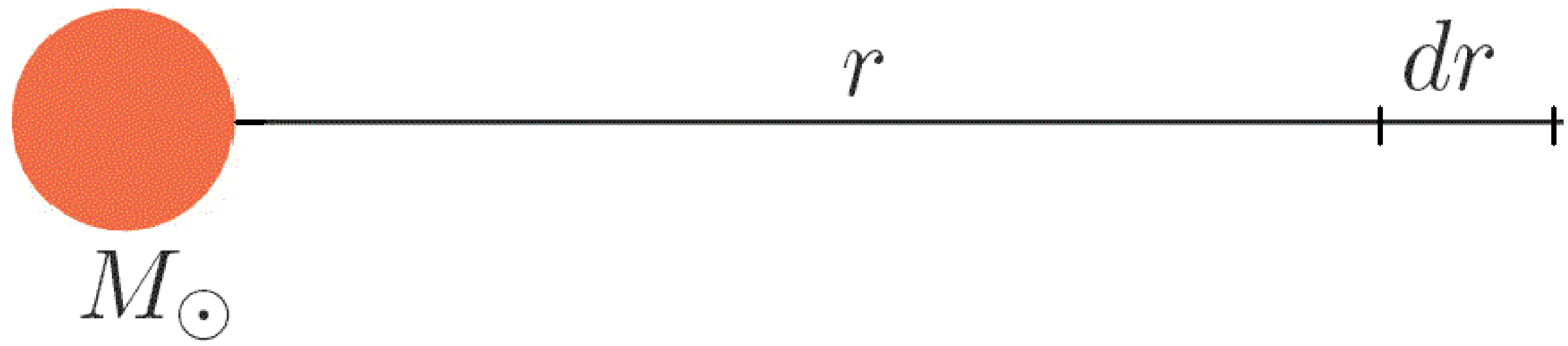
the distribution of asteroids is strongly linked to mean motion resonances

Three–Body Problem

Sphere of action and Hill's radius, R_H

Consider the acceleration due to the Sun at a distance r:

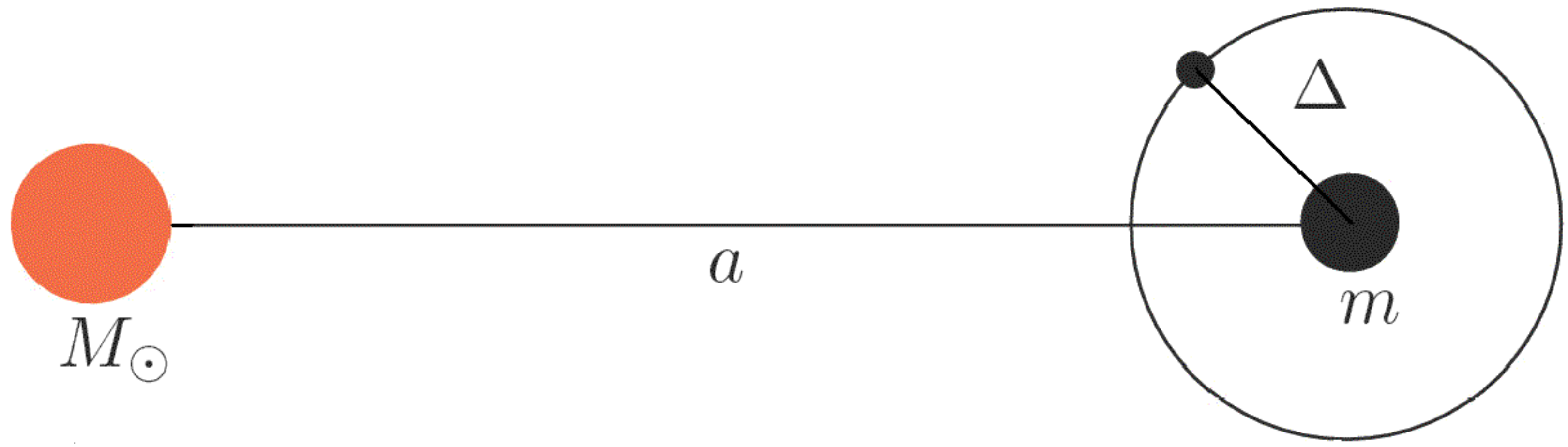
$$\alpha = GM_\odot / r^2$$



A small departure dr generates a variation (a tide):

$$d\alpha = 2GM_\odot \frac{1}{r^3} dr$$

Now consider a satellite orbiting a planet with mass m at a distance Δ.



Taking $dr = \Delta$ the difference in the acceleration between planet located at a mean heliocentric distance a and the satellite is

$$d\alpha = 2GM_\odot \frac{1}{a^3}\Delta$$

The aceleration due to the planet is

$$\alpha_p = Gm/\Delta^2$$

and **when both are comparable** $(d\alpha = \alpha_p)$ **the satellite loses its planetocentric regime** and that occurs for a limit value Δ_L:

$$\Delta_L \sim a\Big(\frac{m}{2M_\odot}\Big)^{1/3}$$

Outside this sphere a planet cannot retain a satellite. A more standard parameter is the Hill's radius (derived from the CR3BP):

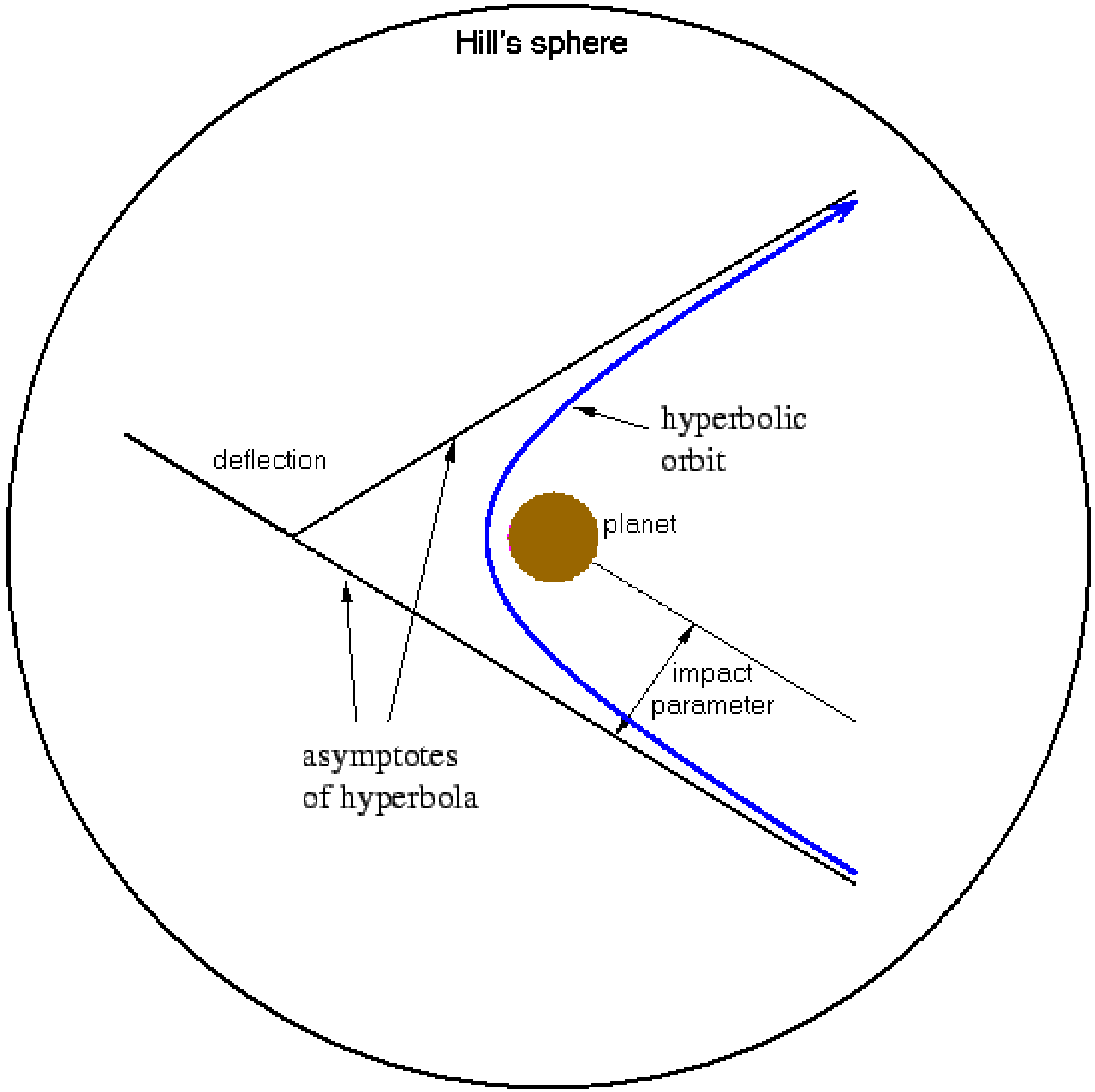
$$\boxed{R_H = a\Big(\frac{m}{3M_\odot}\Big)^{1/3}}$$

It follows that

$$\frac{\text{solar tide}}{\text{planetary acceleration}} = \frac{d\alpha}{\alpha_p} \sim \Big(\frac{\Delta}{R_H}\Big)^3$$

At a distance $\Delta < R_H/4$ for example, the tides due to Sun are negligible. When a spacecraft, comet or asteroid is having a close encounter with a planet **we can neglect the solar perturbation** and consider only the planetocentric orbit of the body (usually an hyperbola) as a first reasonable approximation.

Hyperbolic encounters and impact parameter, σ



Planetocentric orbit

Once an asteroid is well inside the Hill's radius of a planet (for example when $r \sim R_H/4$) we can neglect the perturbations by the Sun and consider only the **hyperbolic planetocentric trajectory** of the asteroid. The planetocentric velocity is

$$v = |\vec{V}_{ast} - \vec{V}_{pla}|$$

and taking $\mu = Gm$ we can write:

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

In general

$$v^2(r \sim R_H/4) \simeq v^2(r = \infty) = -\frac{\mu}{a}$$

Hill's radius is a small distance from an heliocentric point of view but a large distance when observed from a planetocentric point of view.

Then when $r \sim R_H/4$ is reasonable to assume

- two-body problem asteroid-planet
- the asteroid is "at infinity" with respect to the planet

The semimajor axis of the planetocentric hyperbola is

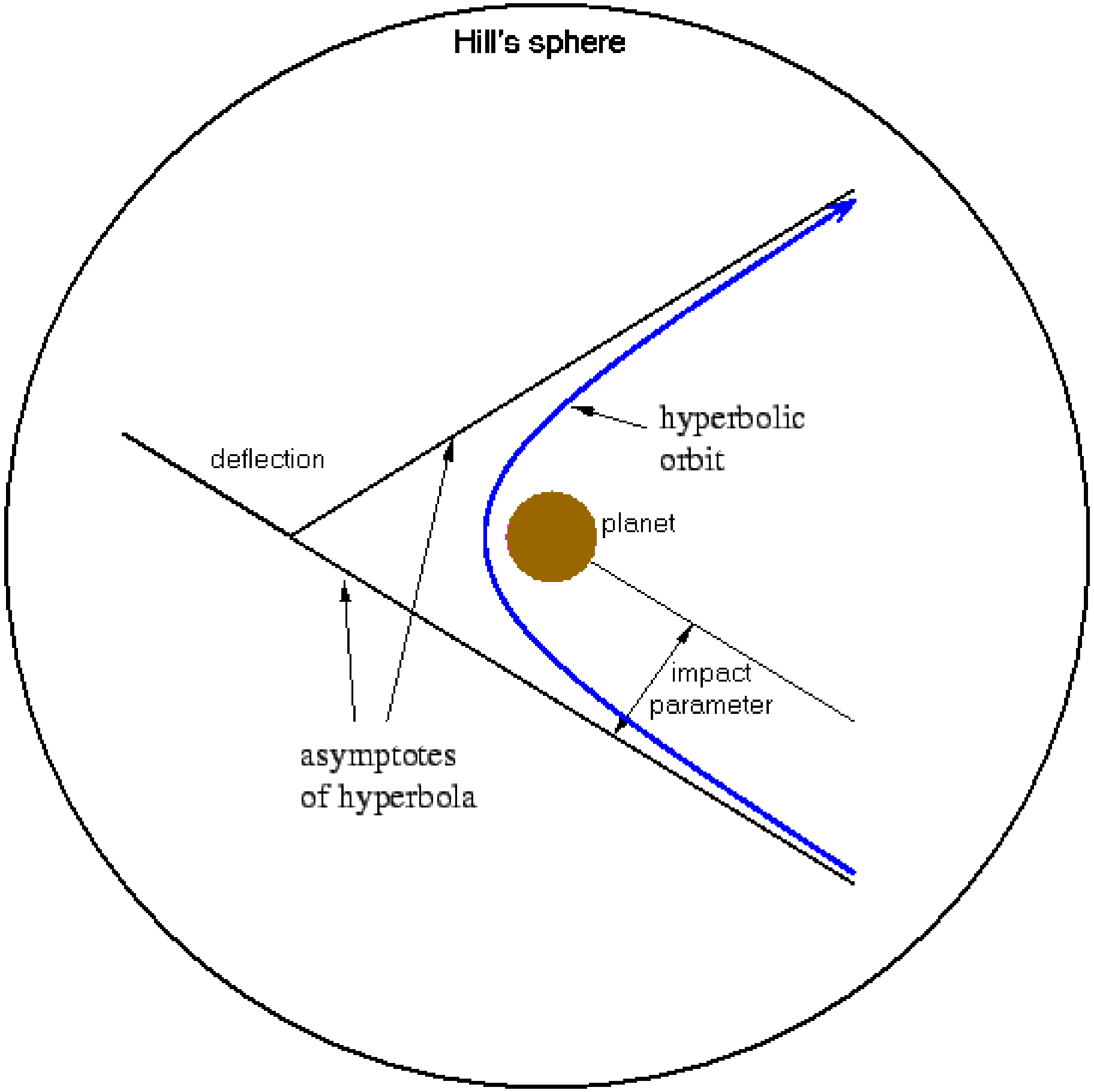
$$a = -\frac{\mu}{v_\infty^2}$$

and the angular momentum is

$$h = \sigma \cdot v_\infty = q \cdot v_q$$

being σ the **impact parameter** which can be related to the **pericentric distance** q of the trajectory:

$$\sigma = q\sqrt{1 - 2a/q} = q\sqrt{1 + \frac{2\mu}{v_\infty^2 q}}$$



A collision with the planet occurs when $q \leq R_{planet}$.

The angular deflection of the planetocentric velocity $\vec{v}$ is γ:

$$\sin\frac{\gamma}{2} = (1 + \sigma v_\infty^2/\mu)^{-1}$$

it can be deduced from the equation of the conic when the true anomaly tends to infinity

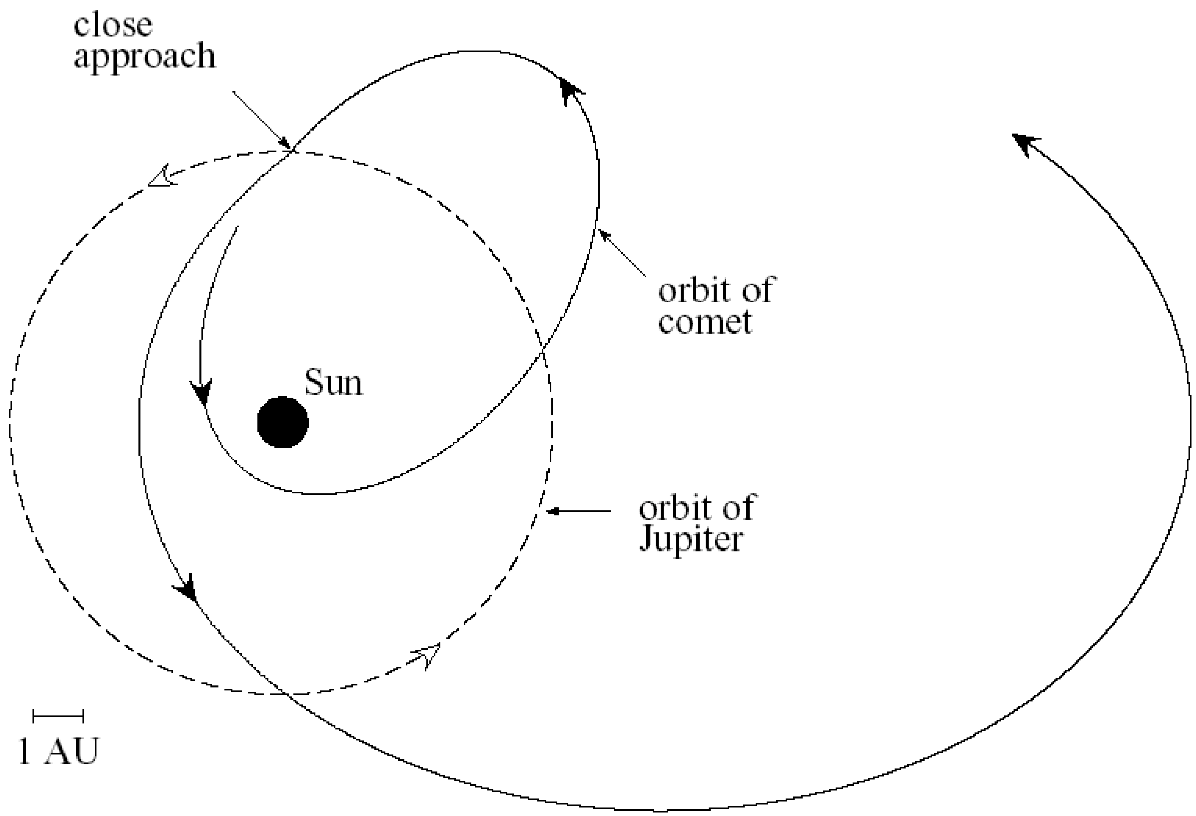
$$\cos f_\infty = -1/e = -\sin(\gamma/2)$$

The new heliocentric orbit

After a close approach the planetocentric velocity of the asteroid rotates an angle γ and the new heliocentric velocity will be different

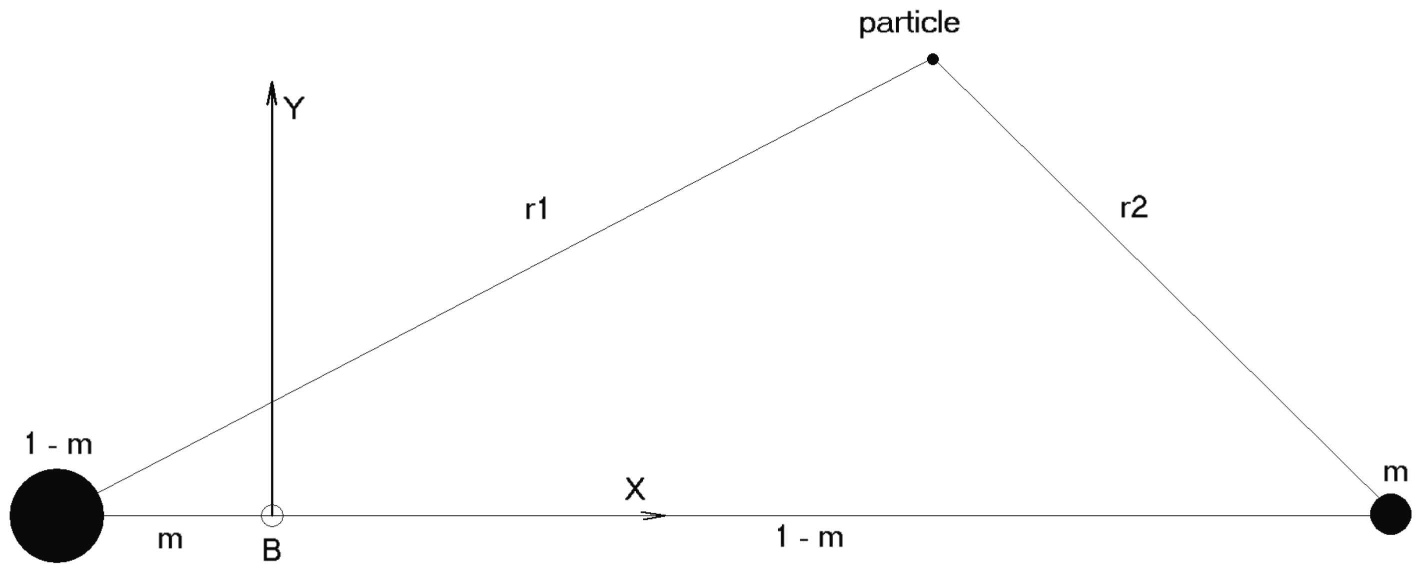
$$\overrightarrow{V'}_{ast} = \overrightarrow{V}_{pla} + \overrightarrow{v'}$$

consequently the heliocentric orbit will be different



(from Murray and Dermott)

CR3BP: Jacobi's constant C



Consider a planet revolving a star with circular orbit of radius a. We redefine units of length, mass and time such that:

$a = 1$ (my new unit of length)

$1 - m$: mass of star

m: mass planet

$G = 1$

the mean motion is $n^2 = G(1-m+m)/a^3 = 1$ rads per unith of time, but

$$n = 1 = \frac{2\pi}{P}$$

then using these units the orbital period of the planet is $P = 2\pi$. The circular velocity of the planet around the star is $V_p = na = 1$.

We define the system $(\hat{x}, \hat{y}, \hat{z})$ which rotates with the planet around the baricenter of the system with angular velocity

$$\vec{\omega} = n\hat{z} = 1\hat{z}$$

Consider a particle located in $\vec{r} = (x, y, z)$. We can demonstrate (see Appendix) the Jacobi's integral of motion of the particle where v is the particle's velocity in the rotating frame:

$$\boxed{C = x^2 + y^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - v^2}$$

C is a constant in the CR3BP. If planet's eccentricity is different from zero C will oscillate around a mean value.

Zero velocity curves

Motion must verify

$$v^2 = x^2 + y^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C \geq 0$$

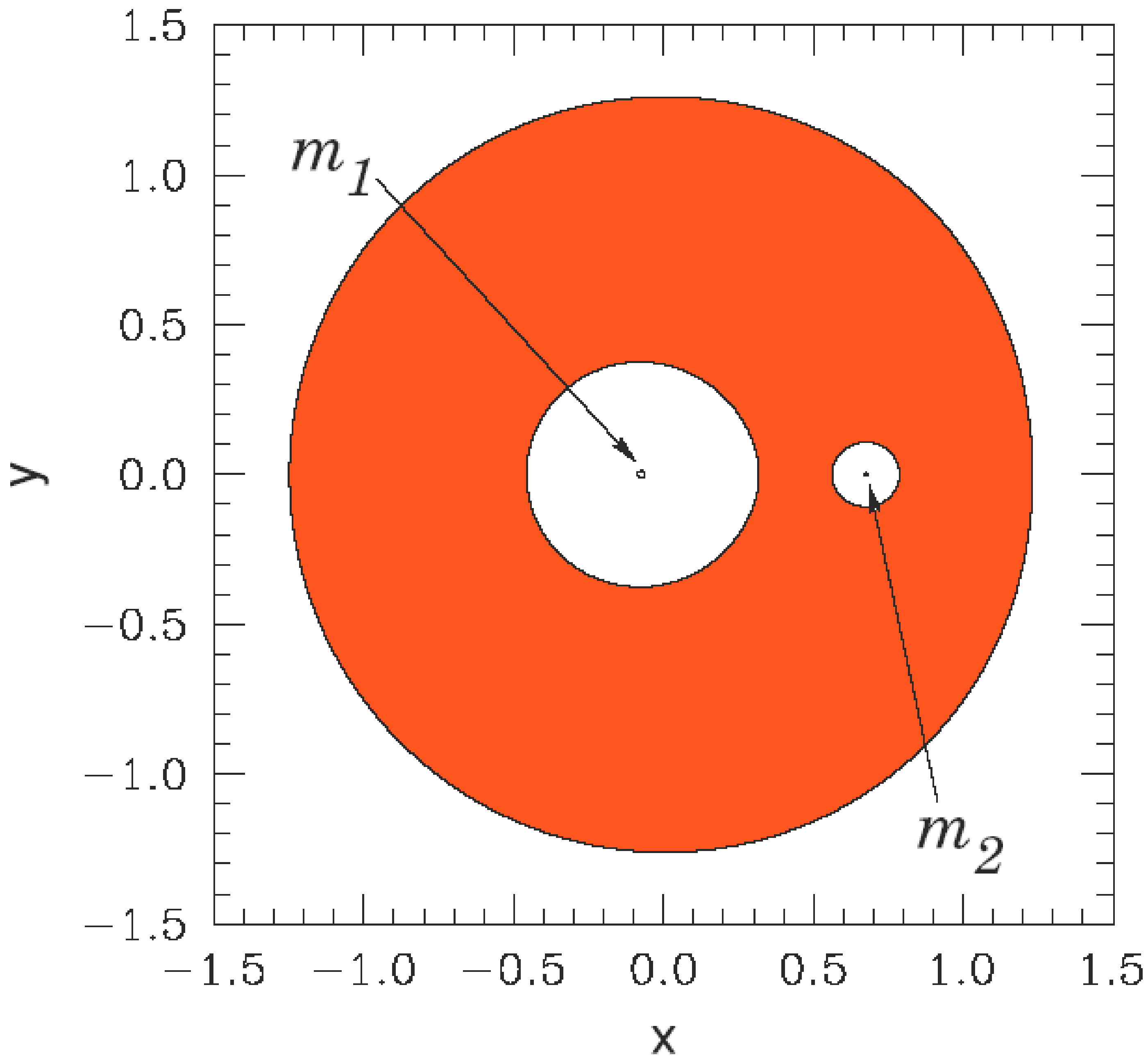
then, the surfaces

$$x^2 + y^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C = 0$$

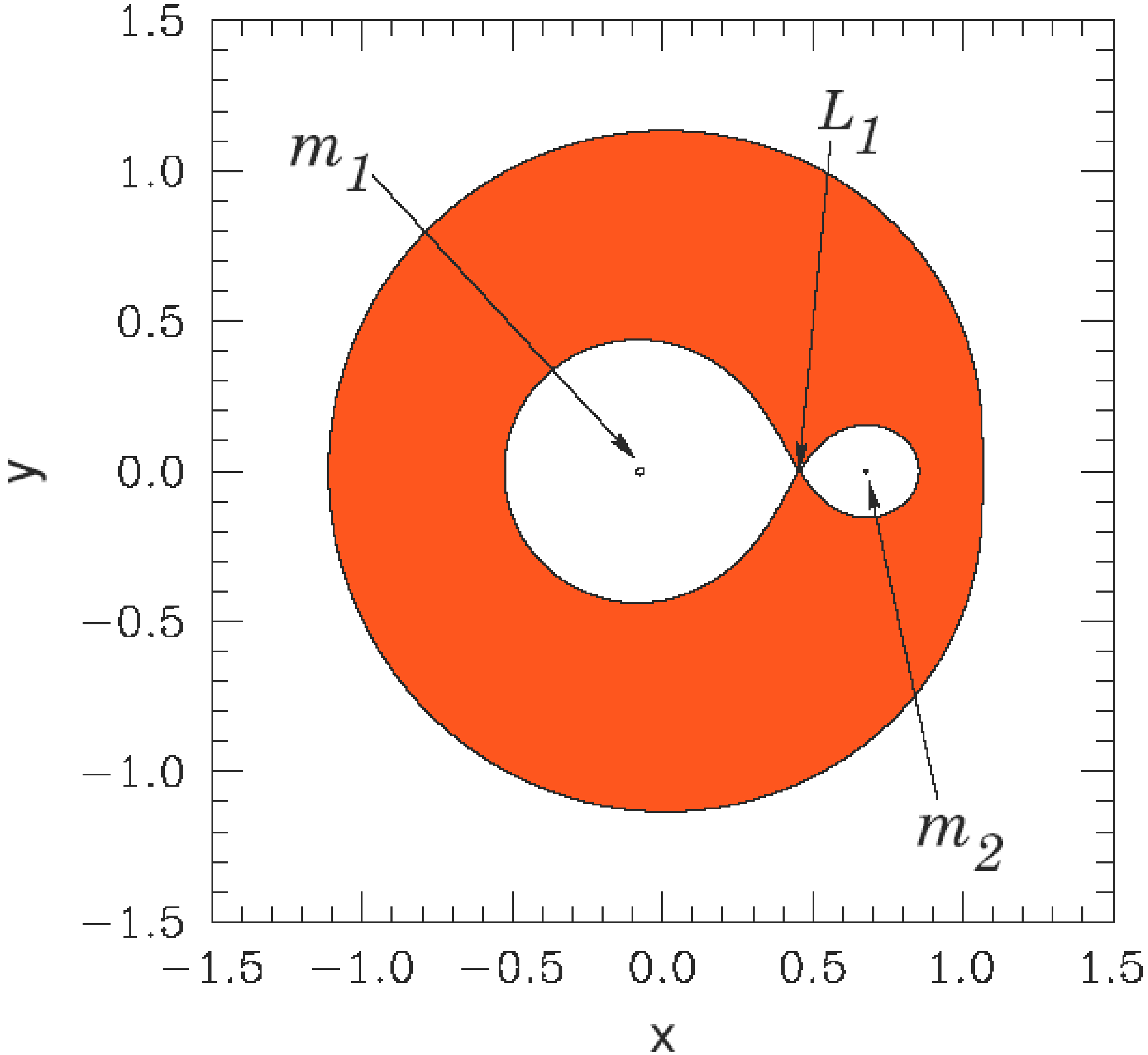
define regions where the motion is confined. These are the **zero velocity surfaces**. They cannot be crossed. We do not have an analytical solution to the problem but we can determine regions where the motion is allowed.

Given some initial conditions, the constant C is defined and the zero velocity surfaces are determined. We can explore these surfaces intersecting with the orbital plane of the planet ($z = 0$).

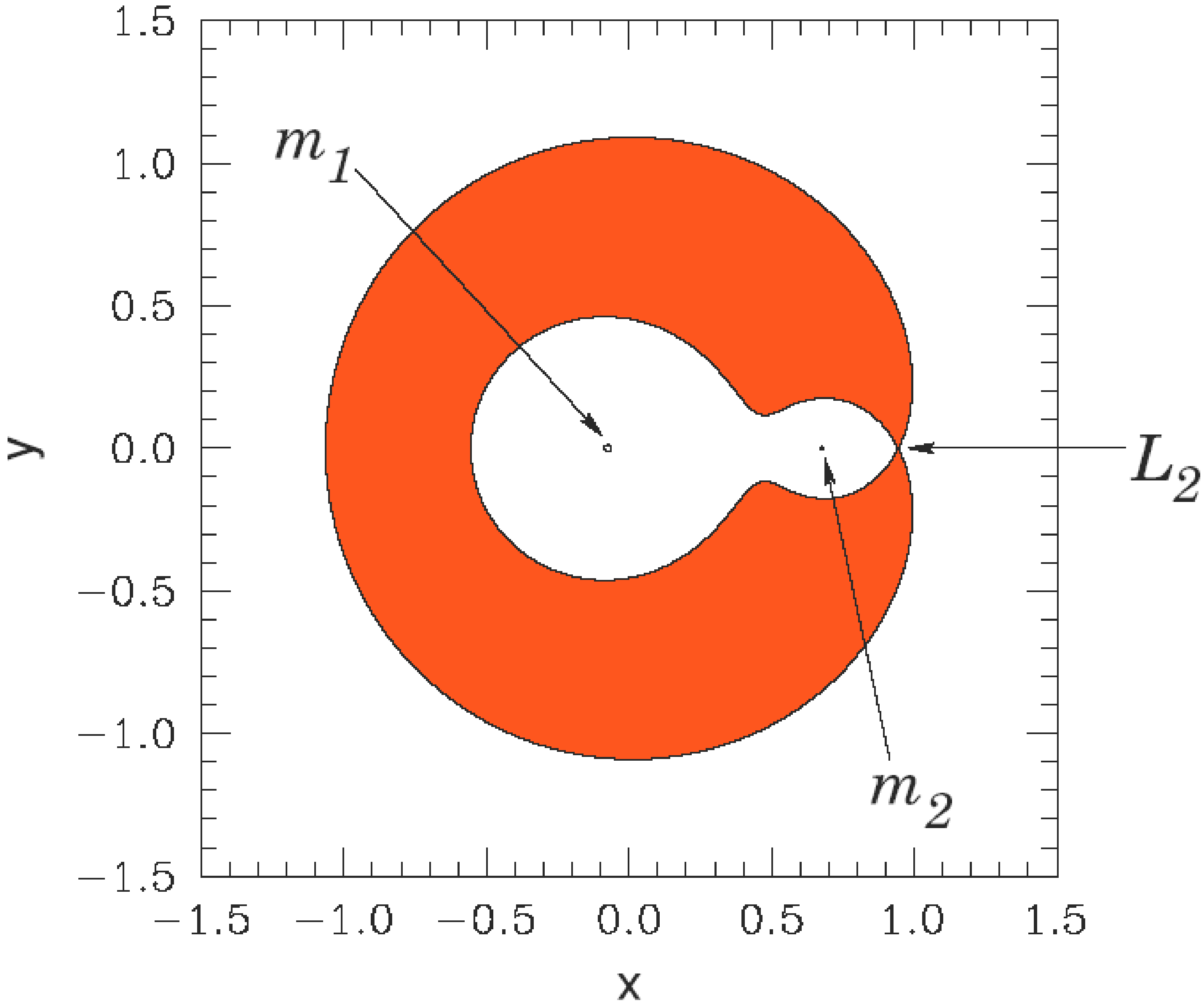
We start with a particle having high C, that means $r_1 \sim 0$, $r_2 \sim 0$ or $x^2 + y^2 \to \infty$.



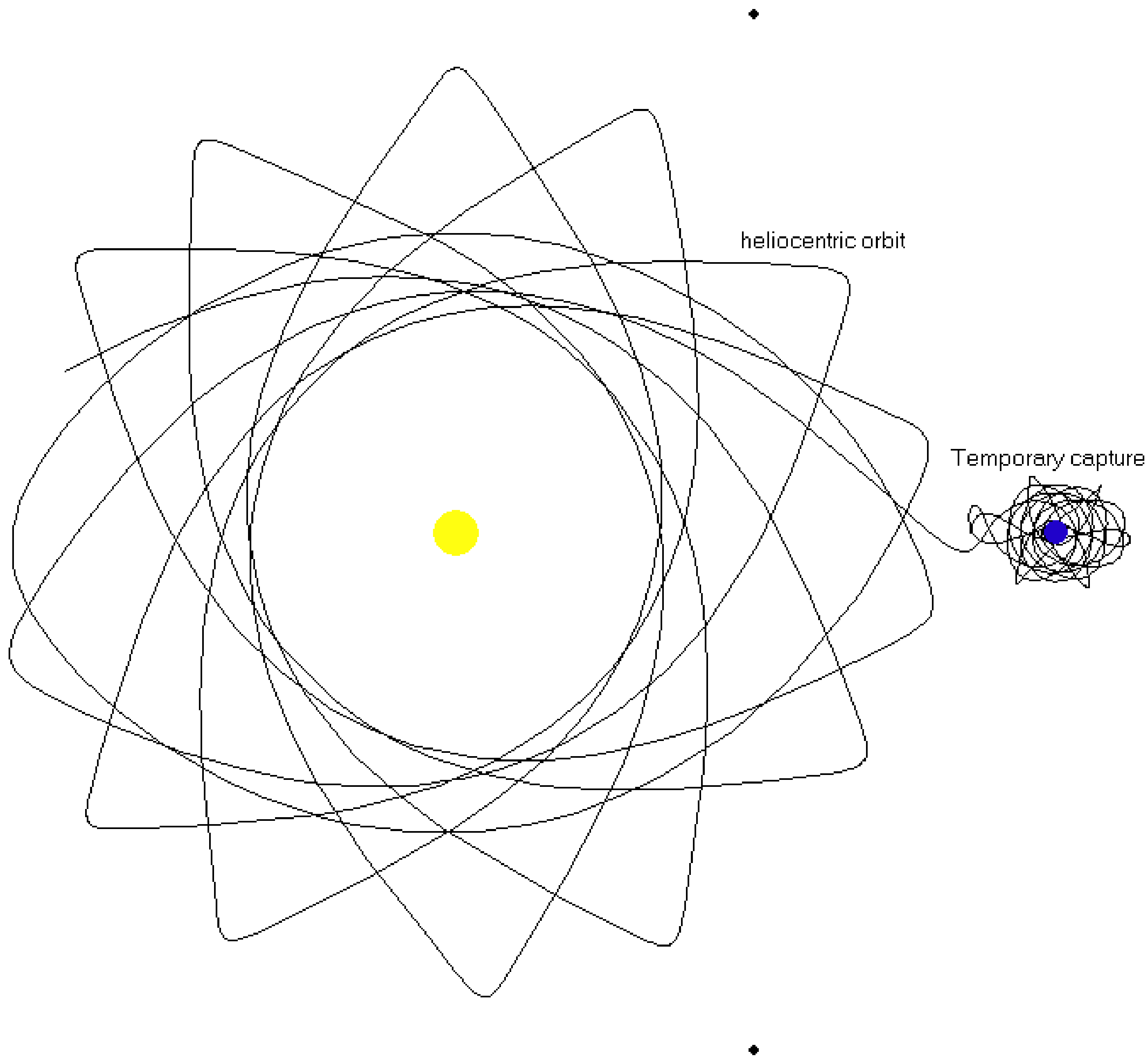
Surface $(v^2 = 0) \cap (z = 0)$ plane. In red, the not allowed region $(v^2 < 0)$.



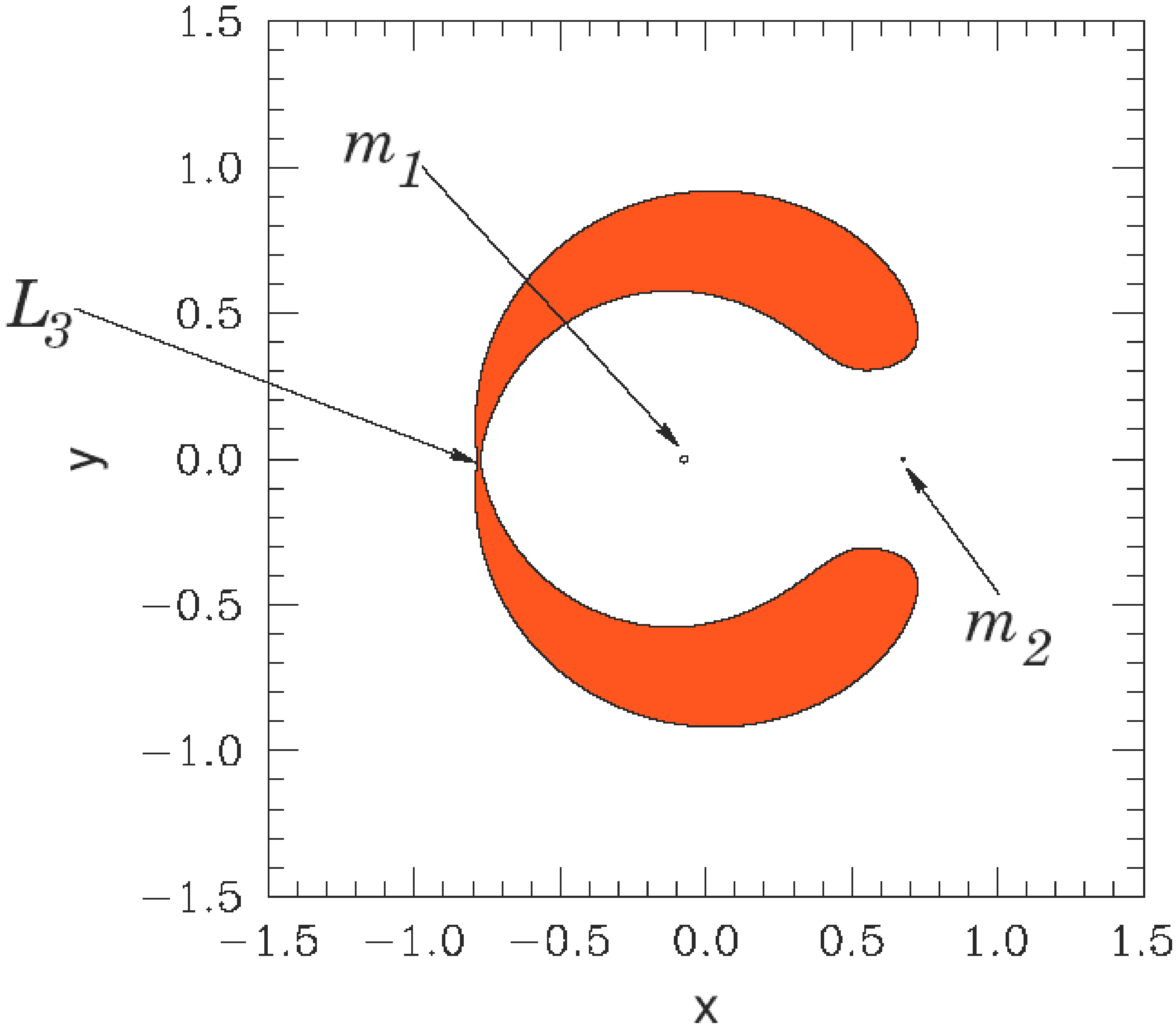
for a lower C it appears the first (unstable) equilibrium point



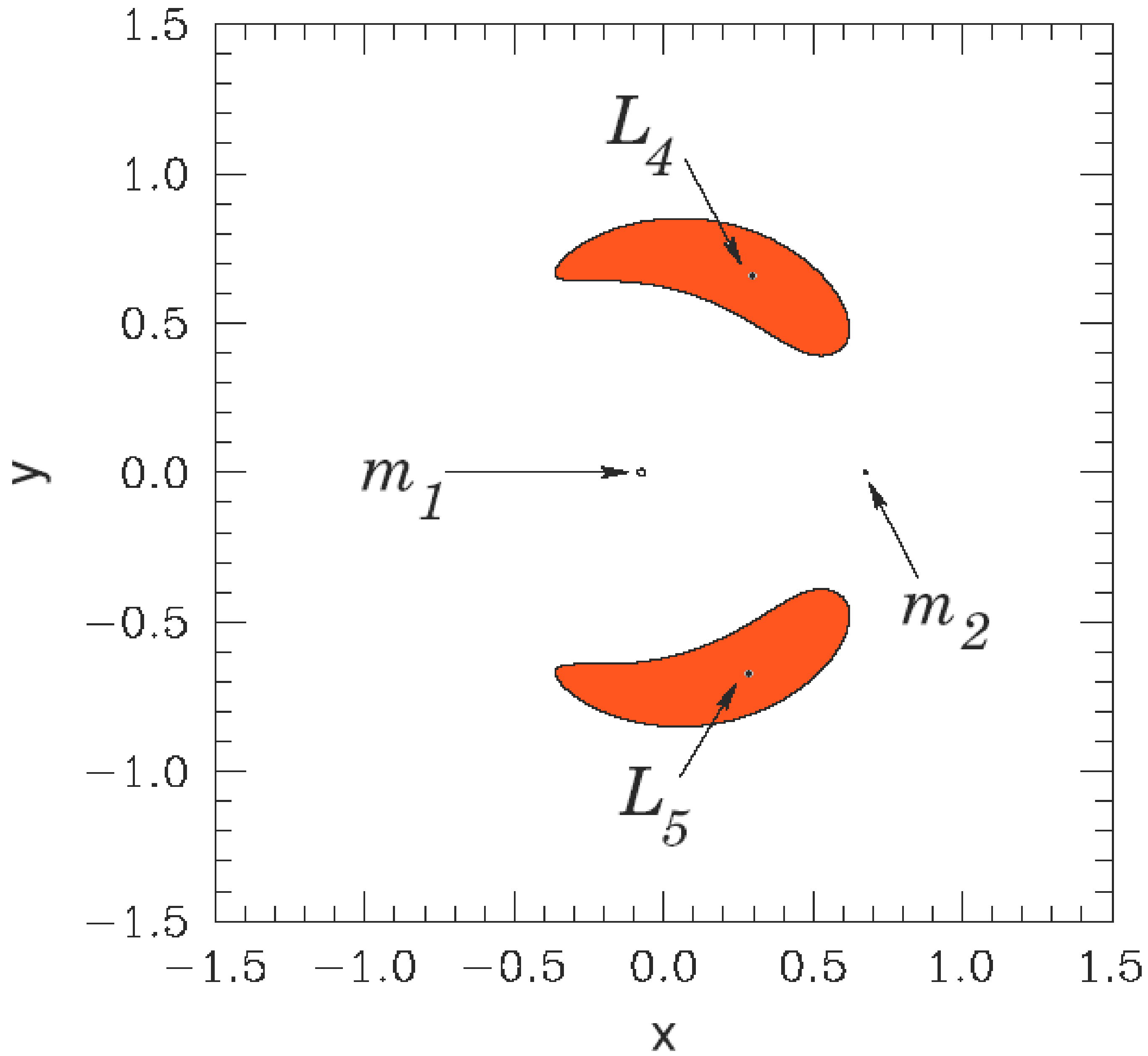
then, the second (unstable) equilibrium point



For some C values nice temporary captures can occur



...and the third (sorry, also unstable) equilibrium point



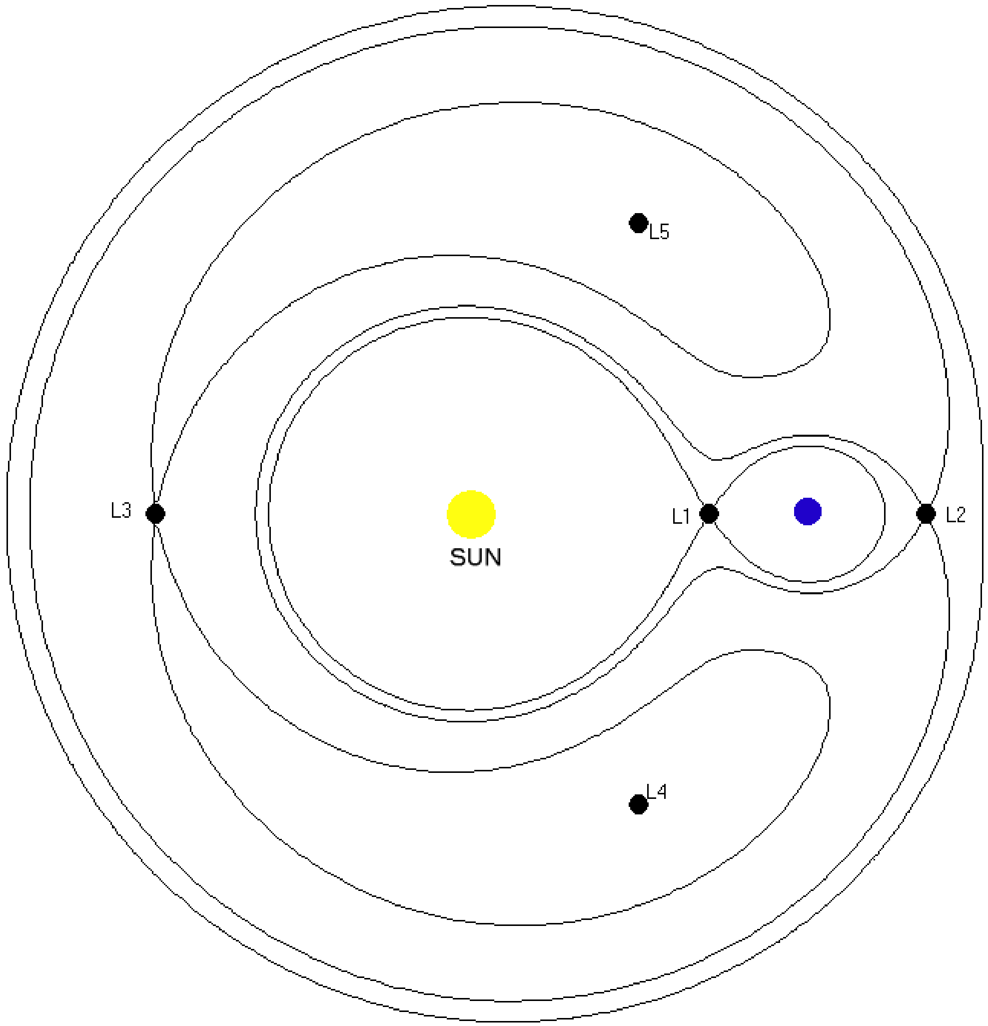
Finally, the stable ones appear when the zero velocity curves collapse into points

Equilibrium points

They are obtained looking for

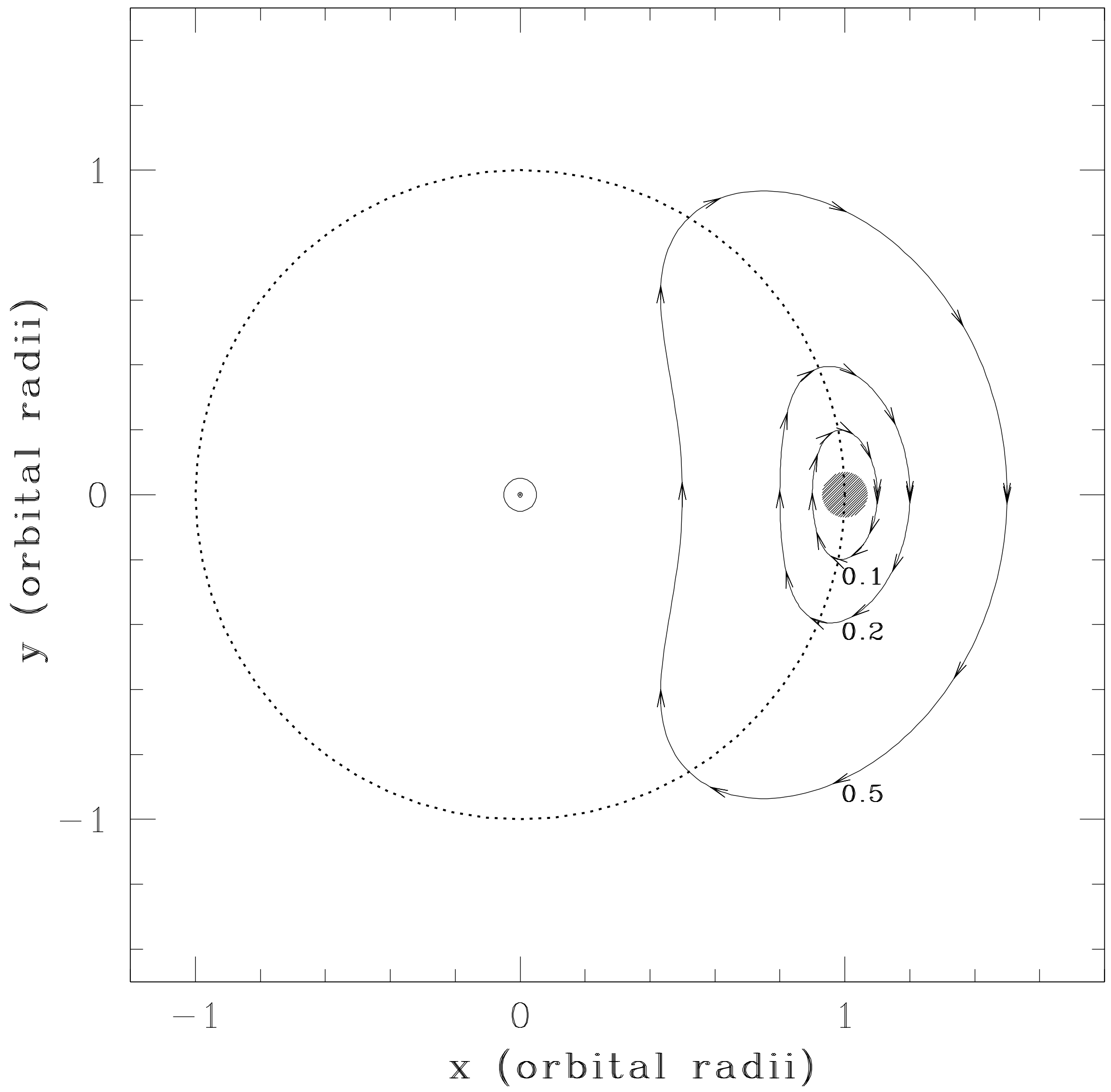
$$\ddot{x} = 0$$
$$\ddot{y} = 0$$
$$\ddot{z} = 0$$
$$v = 0$$

There are only five and all them in the plane $z = 0$.



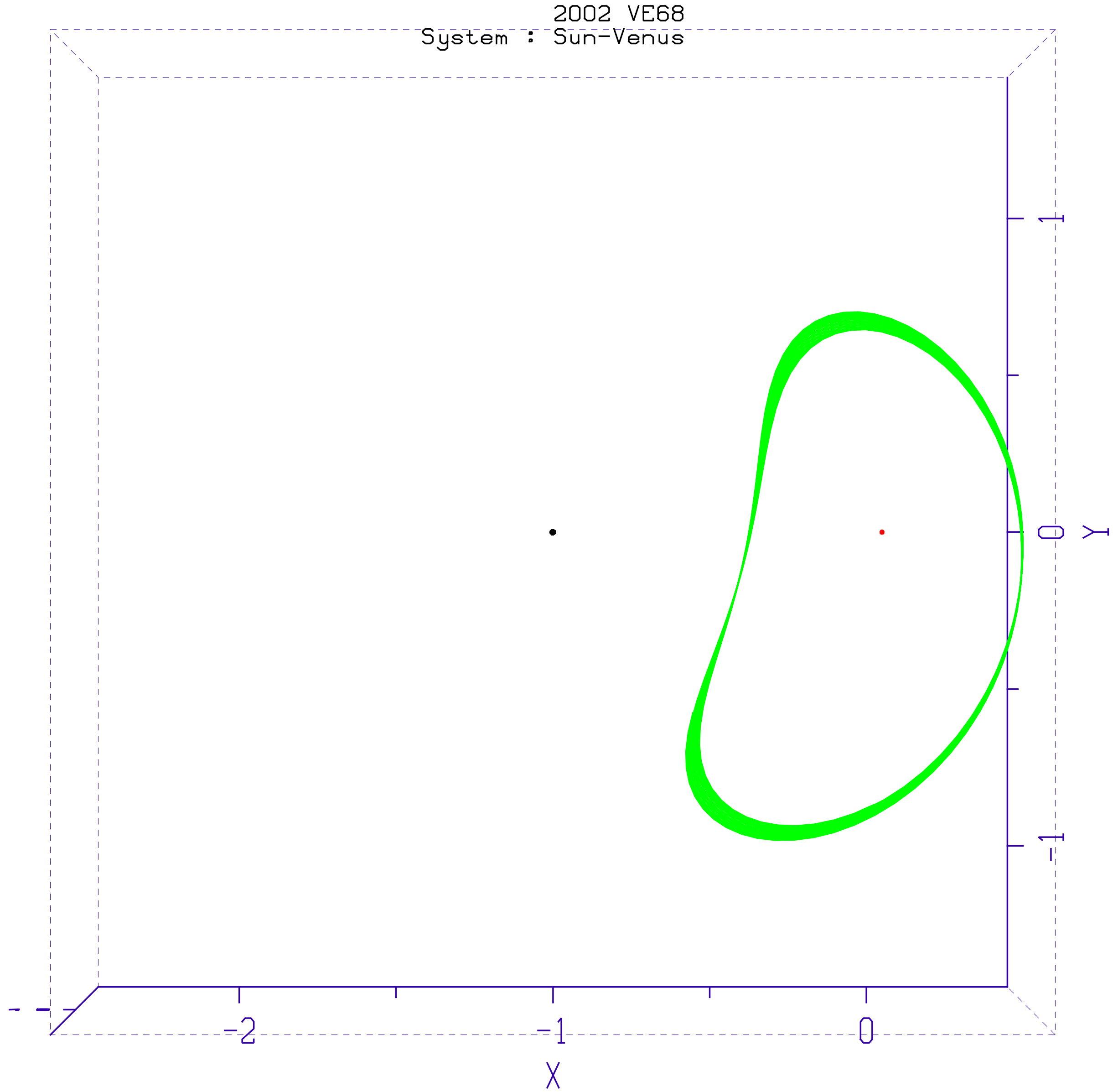
It can be showed that L_1, L_2, L_3, L_4, L_5 are the only equilibrium points and only L_4, L_5 are stable. The distance from $L1$ and $L2$ to the planet is the Hill's radius R_H: the farthest distance allowed for a permanent satellite.

Quasi satellites



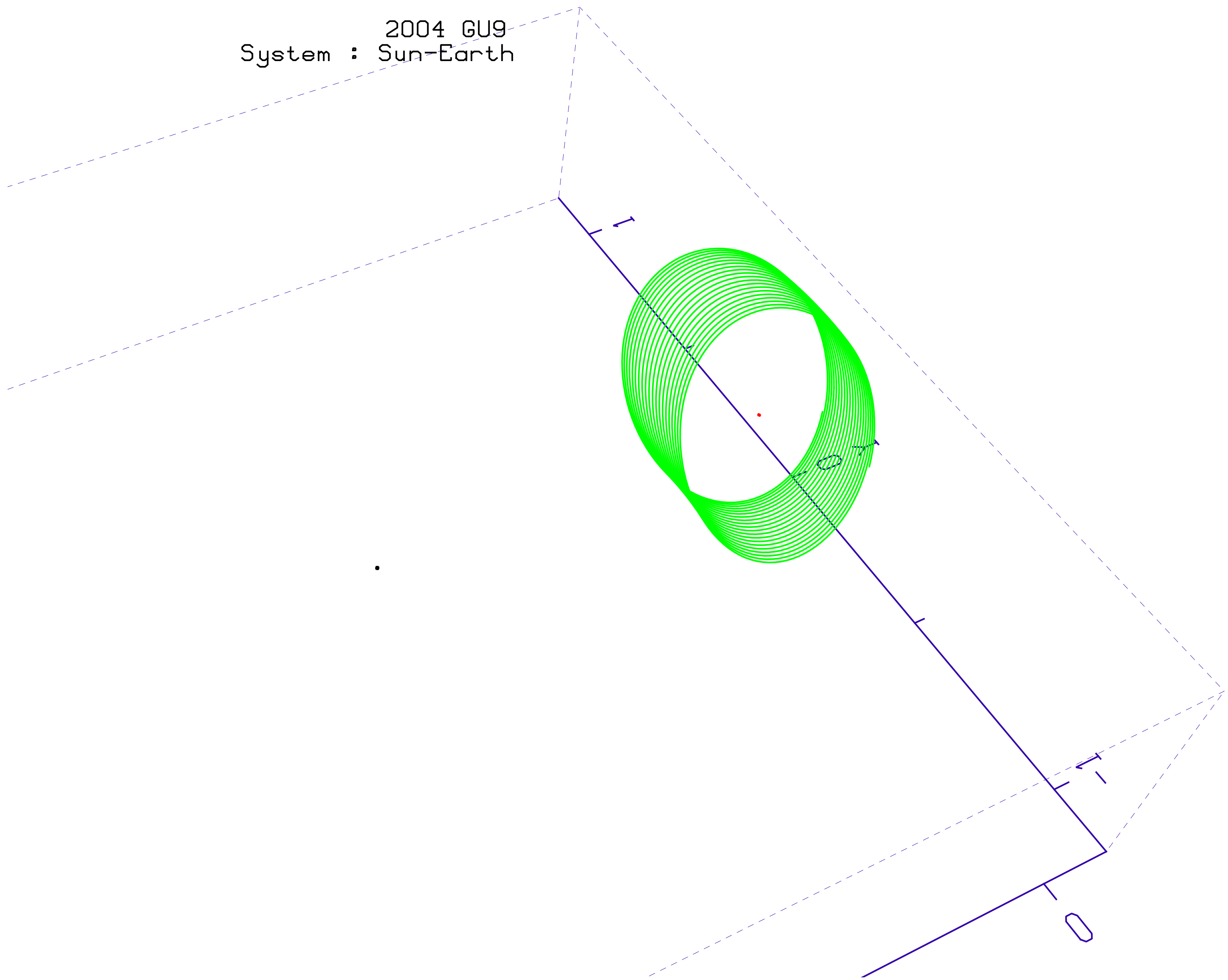
(from Wiegert et al. 2000)

Quasi satellite of Venus



2002VE68

Quasi satellite of Earth

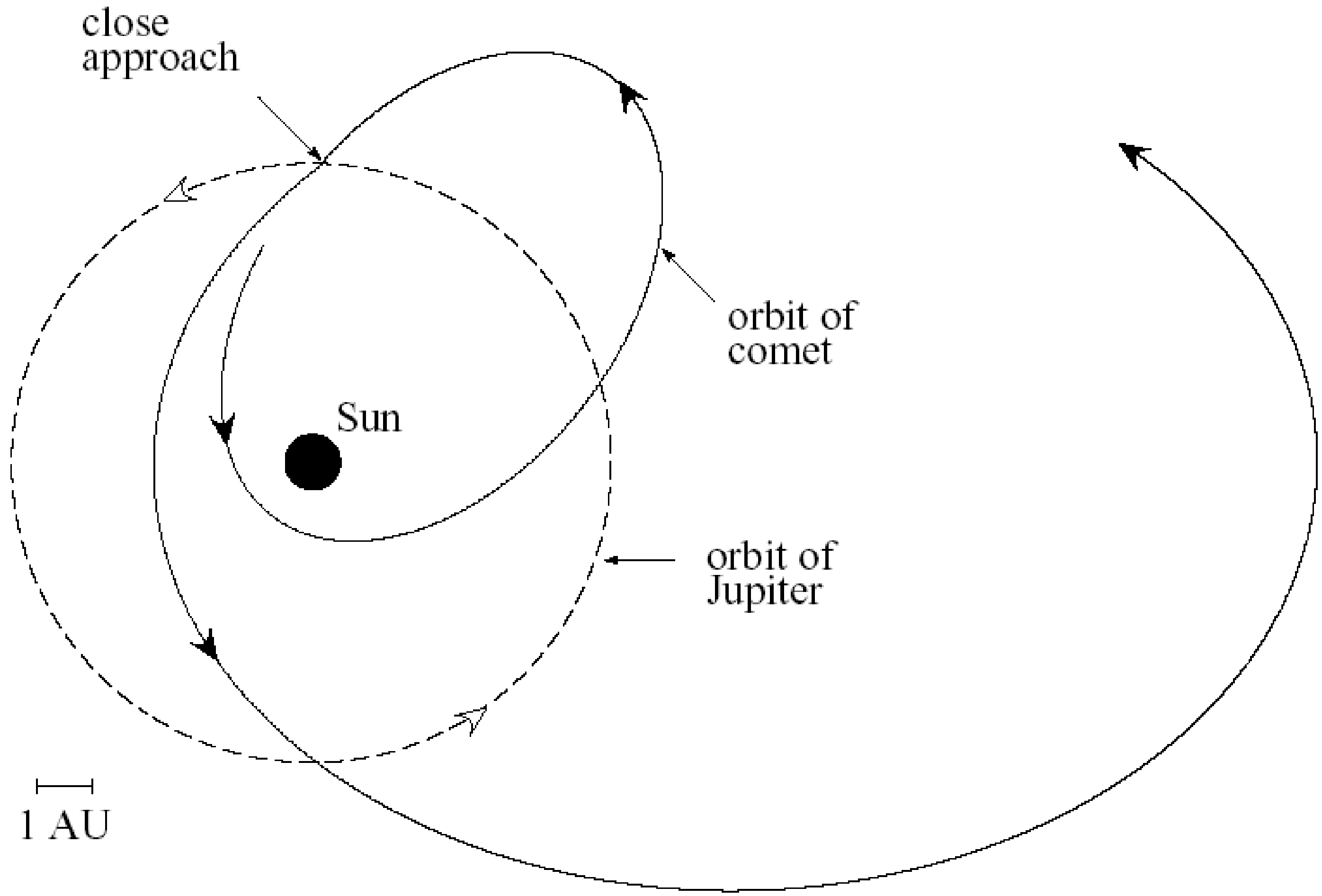


Tisserand parameter, T

It is possible to write Jacobi's integral $C(v, x, y, z)$ as a function of the osculating heliocentric orbital elements of the particle (see Appendix):

$$\boxed{C \simeq \frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i = T}$$

expression which is valid if the particle is far from the Sun and from the planet and taking into account that $m < 10^{-3}$. T is known as the Tisserand parameter. In the CR3BP, C is constant and T presents some departures *only if the orbital elements are determined when the conditions above are not satisfied* (near the Sun or the planet). T should be considered as a simple and approximate form of calculating C.



(from Murray and Dermott)

After a close encounter with a planet the orbital elements (a, e, i) will change but C and T are conserved.

The encounter velocity, U

Suppose the particle is near the planet (then $r_1 \simeq 1$ and $x^2 + y^2 \simeq 1$) but far enough that we can neglect its gravitational attraction (say $r \sim R_H$, where $m/r_2 \simeq 0$) so the particle is "at infinity". Then from Jacobi's integral:

$$v^2 = x^2 + y^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$$

$$v_\infty^2 \simeq 1 + 2 + 0 - T$$

then, under the hypothesis above, the planetocentric velocity "at infinity" of the particle is

$$\boxed{v_\infty \simeq \sqrt{3-T} = U}$$

U is the encounter velocity with the planet **before** the gravitational attraction is felt by the particle (that means "at infinity").

U and T are constant

The orbital elements (a, e, i) can evolve but T and U remain constant, only the orientation of $\vec{U}$ is modified (U rotates γ after the encounter).

It follows that when $T > 3$ encounters cannot exist.

When $T < 3$ they could exist but they are not guaranteed. For example: $a = 2, e = 0, i = 90^o$ implies $T = 0.5$ but the particle never approaches the planet.

If $U \sim 0$ ($T \sim 3$) the planetocentric orbit is quasi-parabolic and a temporary capture by the planet is possible because a slight perturbation can transform the parabola into an ellipse. Then, objects with $T \sim 3$ can experience temporary captures by the planet.

The greatest heliocentric velocity the particle can get after the encounter is (assuming $\vec{U}//\vec{V_p}$):

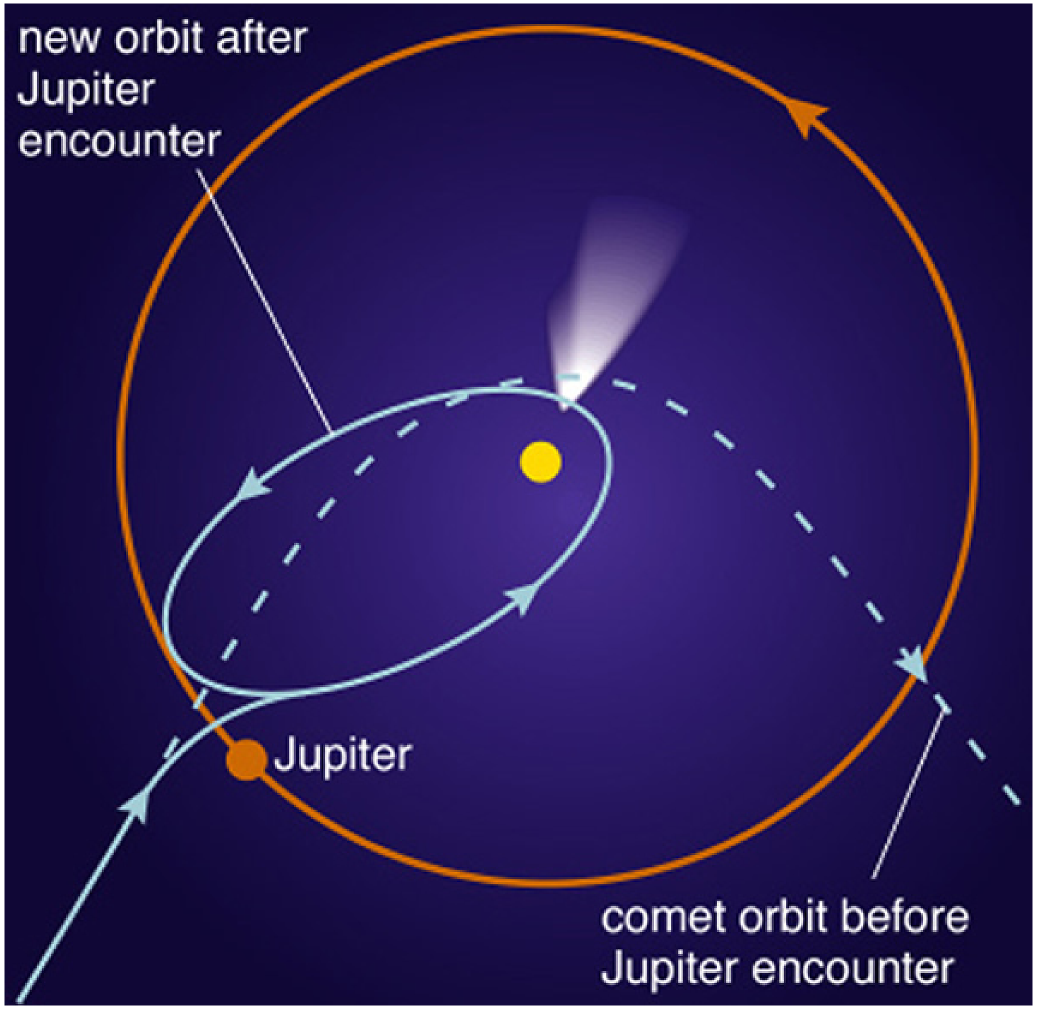
$$V_p + U = 1 + U$$

The escape velocity from the system is $\sqrt{2}V_p = \sqrt{2}$, so if $U \geq \sqrt{2} - 1$ $(T < 2.83)$ the particle eventually could escape from the solar system and conversely if $U < \sqrt{2} - 1$ the particle will never left the solar system by this mechanism. Note that only prograde orbits have $U < 1$.

$T > 3$ no encounters $(v_\infty^2 < 0)$

$T \sim 3$ temporary captures are posible $(v_\infty^2 \sim 0)$

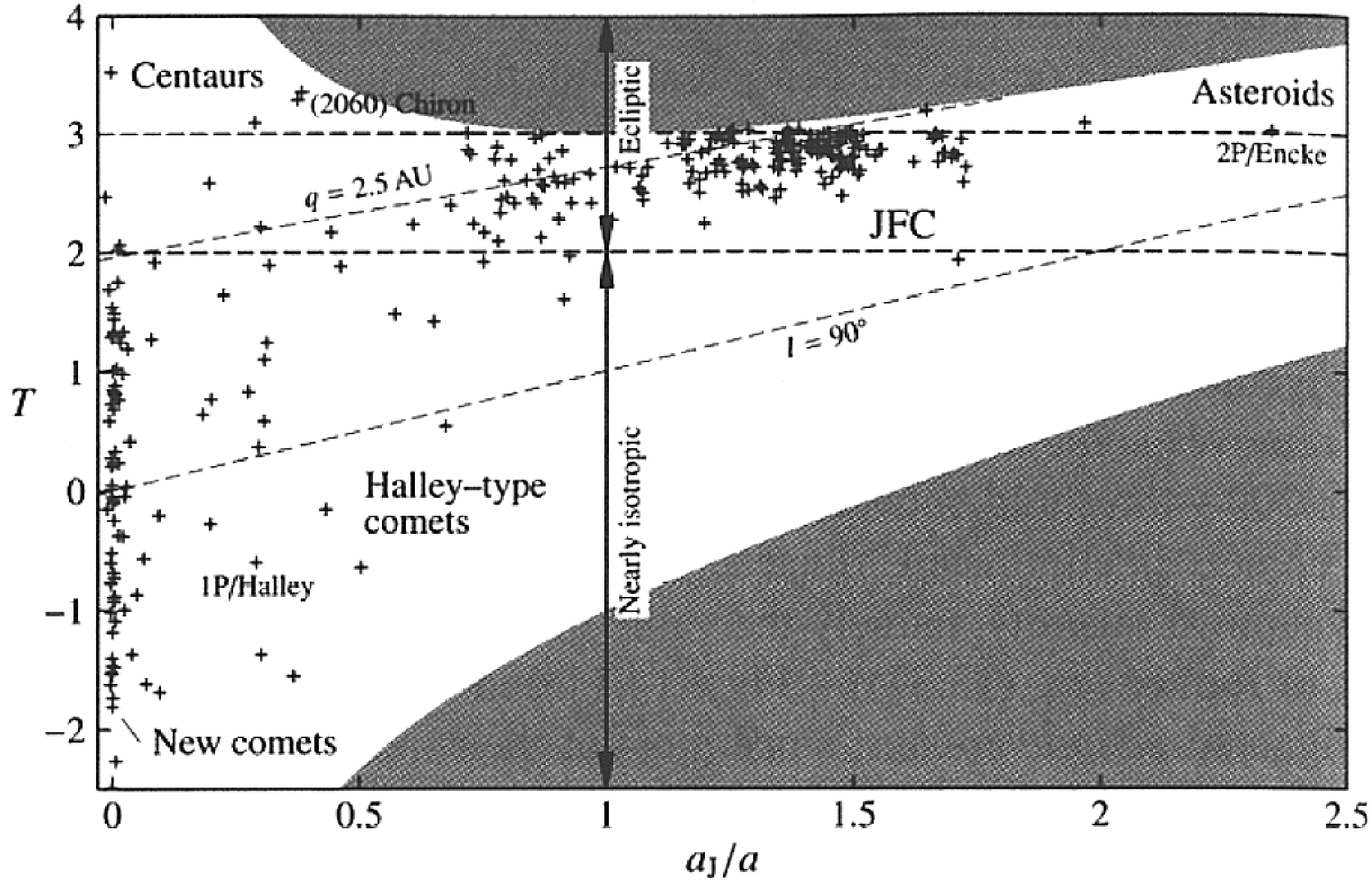
$T < 2.83$ ejection from the solar system is possible, and also captures of new comets



(from Bennet et al.)

This situation theoretically only happen for objects with $T < 2.83$. But if $T << 2.83$ the encounter velocity U will be enormous and the deflection angle γ very small, in consequence the final heliocentric velocity and the orbit will be slightly modified and the capture will not occur.

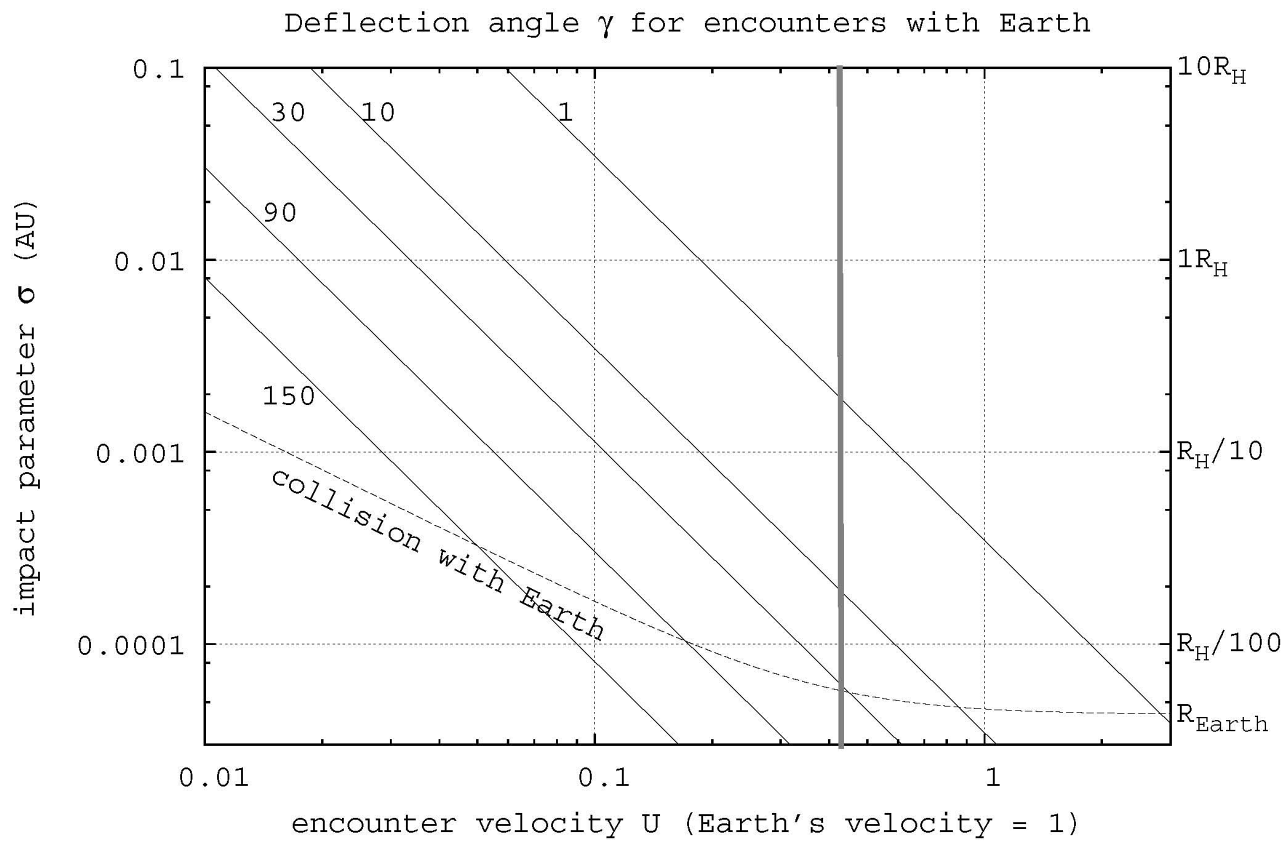
T is a good parameter for classification of small bodies.

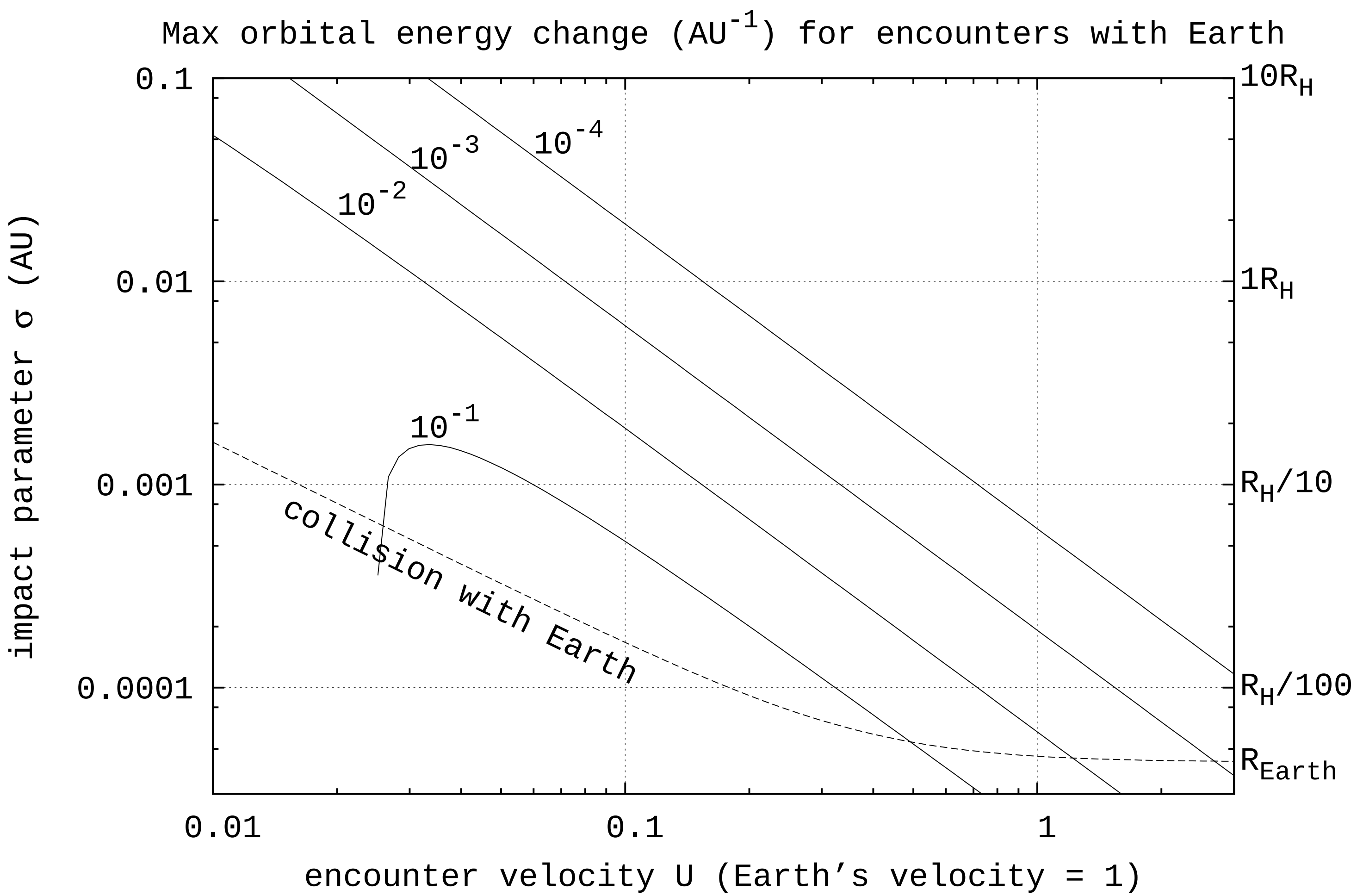


(from Bertotti et al. 2003)

The extreme heliocentric velocity $1+U$ can only be reached if the adequate deflection angle γ is provided so that $\vec{U}//\vec{V}_p$. The deflection angle cannot have an arbitrary value and is defined by σ and U:

$$\tan\frac{\gamma}{2} = \frac{1}{\sigma U^2}$$





see Appendix for details

Appendix

Appendix 1: Jacobi's integral

Demonstration:

The velocity in the inertial frame $\vec{V}$ and the one in the rotating frame $\dot{\vec{r}}$ are related by

$\vec{V} = \dot{\vec{r}} + \vec{\omega} \wedge \vec{r}$

The inertial acceleration is $\vec{\alpha} = -\nabla \mathbf{V}$ where

$$\mathbf{V}(\vec{r}) = -(1-m)/r_1 - m/r_2$$

is the gravitational potential generated by the two masses.

The rotating system rotates with $\vec{\omega} = \hat{z}$ then the relationship between inertial acceleration $\vec{\alpha}$ and the acceleration relative to the rotating system $\ddot{\vec{r}}$ is

$$\vec{\alpha} = \ddot{\vec{r}} + 2\hat{z} \wedge \dot{\vec{r}} + \hat{z} \wedge (\hat{z} \wedge \vec{r})$$

but

$$\vec{r} = z\hat{z} + \vec{\rho}$$

being

$$\vec{\rho} = (x, y, 0)$$

then

$$\vec{\alpha} = \ddot{\vec{r}} + 2\hat{z} \wedge \dot{\vec{r}} - \vec{\rho}$$

multiply by $\dot{\vec{r}}$:

$$\vec{\alpha} \cdot \dot{\vec{r}} = \left[\ddot{\vec{r}} \cdot \dot{\vec{r}} - \vec{\rho} \cdot \dot{\vec{\rho}}\right]$$

then

$$\vec{\alpha} \cdot d\vec{r} = -\nabla \mathbf{V} d\vec{r} = \left[\ddot{\vec{r}} \cdot \dot{\vec{r}} - \vec{\rho} \cdot \dot{\vec{\rho}}\right] dt$$

integrating

$$-2\mathbf{V}(\vec{r}) = \dot{\vec{r}}^2 - (x^2 + y^2) + C$$

or

$$v^2 = x^2 + y^2 - 2\mathbf{V}(\vec{r}) - C$$

then, the particle's velocity in the rotating frame becomes

$$v^2 = x^2 + y^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$$

Appendix 2: Tisserand

The particle has some orbital elements (a, e, i) and we will make to appear them in Jacobi's integral. We need to express position and velocity in the rotating frame $(\vec{r}, \vec{v})$ as function of position and velocity $\vec{V}$ in the inertial frame.

We have

$$\vec{V} = \dot{\vec{r}} + \vec{\omega} \wedge \vec{r} = \dot{\vec{r}} + \hat{z} \wedge \vec{\rho}$$

Then

$$\dot{\vec{r}} = \vec{V} - \hat{z} \wedge \vec{\rho}$$

squaring

$$v^2 = \vec{V}^2 - 2\vec{V} \cdot (\hat{z} \wedge \vec{\rho}) + \rho^2$$

rearranging

$$v^2 = \vec{V}^2 - 2\hat{z} \cdot (\vec{\rho} \wedge \vec{V}) + \rho^2$$

$$v^2 = \vec{V}^2 - 2\hat{z} \cdot (\vec{r} \wedge \vec{V}) + x^2 + y^2$$

$$\vec{V}^2 - 2\hat{z}\cdot(\vec{r}\wedge\vec{V}) = v^2 - x^2 - y^2 = \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$$

(in a numerical integration it is easier to calculate C using the inertial frame than the rotating one)

According to the two body problem baricenter-particle:

$$V^2 = 2/r - 1/a$$

and

$$\hat{z}\cdot(\vec{r}\wedge\vec{V}) = \hat{z}\cdot\vec{h} = \sqrt{a(1-e^2)}\cos i$$

then

$$\frac{2}{r} - \frac{1}{a} - 2\sqrt{a(1-e^2)}\cos i = \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$$

The orbital elements (a, e, i) are referred to the baricenter of the system Star+planet and the inclination is measured with respect to the orbital plane $\hat{x}\hat{y}$ of the planet. In the case of the solar system $m < 10^{-3}$ so it is possible to assume that (a, e, i) are heliocentric.

If the particle **is not very close to the Sun** we have $r \simeq r_1$ then

$$\frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i = 2m\left[\frac{1}{r_1} - \frac{1}{r_2}\right] + C$$

If the particle is far from the Sun and from the planet and taking into account that $m < 10^{-3}$ we obtain

$$C \simeq \frac{1}{a} + 2\sqrt{a(1-e^2)}\cos i = T$$

For elliptic orbits it is possible to express $T(q, Q, i)$ where q, Q are perihelion and aphelion:

$$T = \frac{2}{q+Q} + 2\sqrt{2qQ/(q+Q)}\cos i$$

Appendix 3: The loss cone

The final heliocentric velocity is a vectorial sum:

$$\vec{V} = \vec{V}_p + \vec{U}'$$

or

$$V^2 = 1 + U^2 + 2U\cos\theta$$

being θ the angle between $\vec{V}_p$ and $\vec{U}'$. If $U > \sqrt{2} - 1$ there exists some θ_∞ so that for $\theta \leq \theta_\infty$ the corresponding V is greater than the ejection velocity. This situation occurs for

$$\cos\theta_\infty = \frac{1 - U^2}{2U}$$

If we can assume that $\vec{U}'$ is randomized (deflection γ is so great that θ can get all values from 0 to π) then the **probability of ejection per encounter** is equal to the probability $P(\theta \leq \theta_\infty)$ and this is equal to the solid angle subtended by θ_∞ over 4π which is equal to

$$P_\infty = P(\theta \le \theta_\infty) = \frac{1}{2}(1-\cos\theta_\infty) = \frac{U^2+2U-1}{4U} \qquad (U > \sqrt{2}-1, \gamma > 90^o)$$

Conversely, a comet in an hyperbolic heliocentric orbit has a probability of being captured after an encounter and is equal to $1 - P_\infty$. These results are only valid for encounters satisfying the conditions $(U > \sqrt{2}-1, \gamma > 90^o)$. These are very strong conditions, for example, a particle encountering the Earth never satisfies them. The P_∞ should be weighted with the probability $P(\gamma \ge 90^o)$ which is very low. Weidenschilling (1975) recalculate this issue obtaining more realistic values for the ejection probability.

Appendix 4: Geometry of encounters

The velocity of encounter U form an angle θ with the planet's heliocentric velocity ($\vec{V}_p = \hat{y}$) and is rotated an azimuthal angle ϕ around $\hat{y}$. Then:

$$U_x = U \sin\theta \sin\phi$$

$$U_y = U \cos\theta$$

$$U_z = U \sin\theta \cos\phi$$

Assuming the asteroid is encountering the planet: $r = 1$ and $V^2 = 2 - 1/a$. The "angular momentum" is

$$\sqrt{a(1-e^2)} = rV_t$$

where V_t is the transverse velocity. In consequence the radial velocity evaluated at $r = 1$ is

$$V_r^2 = V^2 - V_t^2 = 2 - 1/a - a(1-e^2)$$

The encounter with the planet occurs at the line of the nodes of the asteroid's orbit then:

$$V_y = V_t \cos i$$

$$V_z = V_t \sin i$$

$V_x = V_r$

Then the relative velocity

$$\vec{U} = \vec{V} - \vec{V}_p = (V_x, V_y - 1, V_z)$$

has components

$U_x = \pm\sqrt{2 - 1/a - a(1-e^2)}$

minus sign is for encounters at pre perihelion passage ($\dot{r} < 0$)

$U_y = \sqrt{a(1-e^2)}\cos i - 1$

$U_z = \pm\sqrt{a(1-e^2)}\sin i$

minus sign is for encounters at the descending node of the asteroid's orbit ($\dot{z} < 0$)

Conversely

$$a = \frac{1}{1 - U^2 - 2U_y}$$

$$e = \sqrt{U^4 + 4U_y^2 + U_x^2(1 - U^2 - 2U_y) + 4U^2 U_y}$$

$$i = \arctan \frac{U_z}{1 + U_y}$$

or also

$$\sin^2 i = \frac{U_z^2}{U_z^2 + (1 + U_y)^2}$$

If we define the heliocentric "energy" of the particle as $x = 1/a$ then we have

$$x = 1/a = (1 - U^2 - 2U\cos\theta)$$

and the variation in the energy due to the encounter is

$$\Delta x = 1/a' - 1/a = 2U(\cos\theta - \cos\theta')$$

Maximum variations in energy are:

$$\Delta x = 2U(1 - \cos\gamma(\sigma, U))$$